UNIT-2

GRAPH THEORY

Lesson Structure

2.0 Objective

Graphs are discerte structures consting of vertices and edges connecting these vertices. The objective of study of graph lies in its applicability to some problem in almost every conceivable discipline using graph models. It has made its uses felt not only in mathematical field only but also in diverse fields like economics, biology, psychology, Computers, bridge problem and puzzles. Graphs can be used to study the structures of the world wide web.

2.1 Introduction

The concept of graph that we are going to study here is quite different from the kind of graph we have already dealt in mathematics. How we are not concerned with the set of points (x, y) as coordinates. The graph which we are going to study here are simple geometrical figures consisting of points and lines connecting some of these points.

The basic idea of graph was introduced in the Eighteenth century by Leonhard Euler (a swiss mathematician) in 1736. At that time graph theory was supposed to be insignificant from mathematics points of veiw. But recent developments in mathematics and its applications have given a strong impetus to graph theory. In nineteenth century graphs were used in such fields as electrical circuitry and moleculor diagrams. In the theory of mathematical relation graph theory is used as a natural tool. Transportation Problem, flow in Pipeline networks. The study of graph theory help to determine whether two computers are connected by communication link using graph models of computer network. Graphs with weights assigned to their edges can be used to some problems such as finding the shortest path between two cities in a transportation network. Graph can also be used to schedule exams and assign channels to television stations.

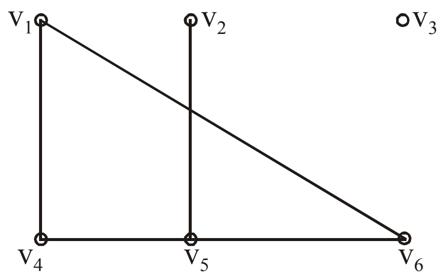
2.2 Definitions

2.20 Definition (Graph) or (Simple path) A graph G is the pair G = (V, E) consisting of vertex set V and the edge set E of unordered pairs of distinct elements of V.

Example – 2.20

Let $V = \{V_1, V_2, V_3, V_4, V_5, V_6\}$

$E = \{V_1V_4, V_1V_6, V_2V_5, V_4V_5, V_5V_6\}$



Then the graph G = (V, E) Can be represented as in this figure

It has six vertices and five edges.

Hence each edge is associated with two verticals

A simple graph is not sufficient to model several networks (viz. multiple telephone lines betweem computer in a network, etc.) In such cases we use multigraphs which uses vertices and undirected edges between these vertices with multiple edges between pairs of vertices allowed.

It can be seen that every simple graph is also multigraph. But not all multigraphs are simple graphs (since in a multigraph two or more edges may connect the some pair of vertices). Hence we cannot use a pair of vertices to specify an edge of a graph when multiple edges are present.

2.21 Definition :

Order of G : It is the number of vertices of G. We denote it by p. q denotes the number of edges of G.

Vw will denote the edge {v, w}. v and w are vertices of G.

2.22 Def. Join. –

An edge of G is called an edge if e = vw (Also e is said to join the vertices v and w)

Two such vertices are called adjacent.

We also call e is incident to v and w and that w is a neighbour of v.

2.23 Def. N(v) –

It denotes the set of all vertices of a adjacent to v.

2.24 Def. Adjacent edges. –

Two edges of a incident to the same vertex are called adjacent edges.

2.25 Def. Hypergraph –

A hypergraph G is a pair (V(G), E(G)) where V(G) is a finite non-empty set of vertices and E (G) is a finite set of unordered sets of distinct elements of V (G) called edges.

Def. 2.26 K-uniform hypergraph. –

If all the edges in E (G) have the same cardinality K.

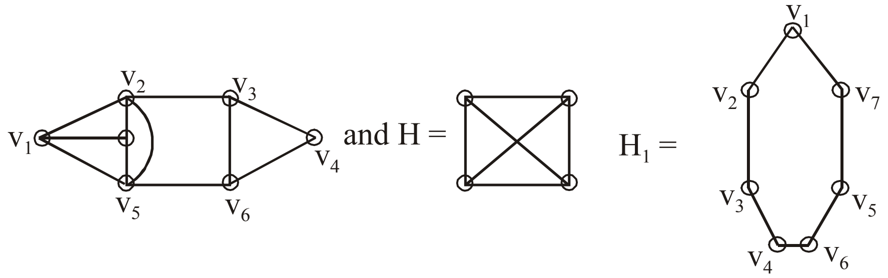
Def 2.27 Degree of vertex –

For each vertex V in a graph G, the number of edges incident to V is called the degree or valency of V. It is denoted by d(V) or ρ(V).

A vertex of degree O is called isolated vertex. A vertex of degree are is called an end vertex. Δ (G) denotes maximum degree in G and δ (G) denotes minimum degree in G.

Def. 2.28 Regular graph –

If all the vertices of G have the same degree, then a is called a regular graph. G is called K-regular if each is of degree K. A O-regular graph is called null-graph. A 3-regular graph is called trivalent graph .(cubic graph).



is also a subgraph of G

Paths –

A sequence of edges of the form $v_0, v_1, v_2, \ldots\ldots, v_{r-1} v_r$ (also abbreviated as $v_0, v_1 \ldots\ldots v_r$) is called a walk of length r form v_0 to v_r. (v_0 is called initial vertex of the walk and v_r is called terminal vertex).

A walk is called a trial if those edges are all distinct.

Def. 2.29

A walk is called a path if the vertices $v_0, v_1, ..., v_r$ are also distinct.

N.B. :

A path is neassarily a trial. The converse may not be true. Two paths in a graphs are called edge-disjoint if they share no common edges. It is called vertex disjoint if they share no common vertices.

A walk or trail is called open if $v_0 \neq v_r$ and closed if $v_0 = v_r$.

Def. 3.29.1

Circuit –

A walk in which the vertices $v_0, v_1, ...v_r$ are all distinct except for v_0 and v_r which coincide is called a circuit (or a closed trial is called a circuit).

A circuit is called even if it has an even number of edges and odd otherwise.

A triangle is circuit of length 3. A Quadilateral is a circuit of length 4. d (v, w) will mean length of shortest path between two vertices v and w of G.

Def. 2.29-2 Cycle –

A circuit which the first vertex appears exactly twice (at the begining and the end) and in which no other vertex appears more than once is a cycle.

An n-cycle is a cycle with

n-vertices. It is even or odd as n is even or odd.

Def. 2.29-03 Connected Graph –

A Graph is conneeted if there is path joining each pair of vertices of G.

It is disconnected if not connected.

2.29-04 Def. Eulerian graph –

A connected graph G is Eulerian if it has a closed trail which includes every edge of E(G). Such a trial is known as Eulearian trial.

Eulerian circuit are named after Leonhard Euler, the solver of konlgsberg Bridge problem.

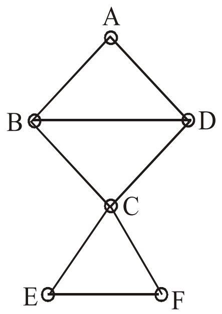
2.29-05

Def : Hamiltonian Graph – A graph G is called Hamiltonian if it has a circuit which includes every vertex of V(G). Such a circuit is called a Hamiltonian circuit.

3.29-06 Def :

Traceable graph. – A graph G is called traceable if it has a path which includes every vertex of V(G). Such a path is called a Hamiltonian path.

2.22 Example

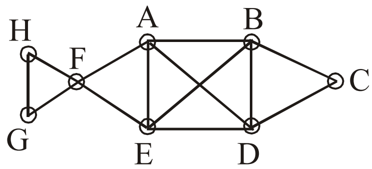


In a graph given below, A B C D E F CBD is a walk of length 7 which is neither a trial nor a path; A B C E F C D is a trial but not a path ; ABCEFCDBA is a closed walk whcih is not a circuit; BCEFCDB is a circuit which is not a cycle and BCDB is a 3-cycle and hence a odd cycle. The closed walk CEFCBDC is not a cycle because the first and last vertex appears a third type.

Example. 2.23

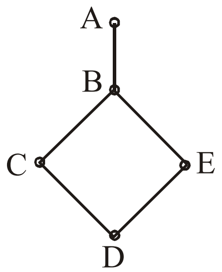
In the above figure, the circuit ABCEFCDA is not Eulerian because it does not contain the edge BD.

Example.2.24



(Demonstrating difference between a circuit and an Eulerian circuit).

consider the graph.



Here the circuit ABCDEFGHFA is not Eulerian because even if it contains all vertices it fails to contain four edges.

However, this graph is Eulerian because ABCDEFGHFADBEA is an Eularian circuit.

This graph is not Eulerian.

2.29-07 Definition Path is a directed graph.

A path form a to be in a directed graph is a Sequence edges (x_0, x_1), (x_1, x_2), ... (x_{n-1}, x_n) in G.

Where n is a non negative integer and $x_0 = a$, $x_n = b$. This means it is a sequence of edges where that terminal vertex of an edge is the same as the initial vertex in the next edge in the path.

We denote the path as $x_0, x_1, \ldots\ldots\ldots\ldots x_{n-1}, x_n$ and call it of length n.

Clearly, an empty set of edges is a path from a to a.

As such , a path of length n $\geq$ 1 that begins and ends at the same vertex is called a circuit or cycle.

It can be seen that a path in a directed graph can pass through a vertex more than once.

Example 2.25

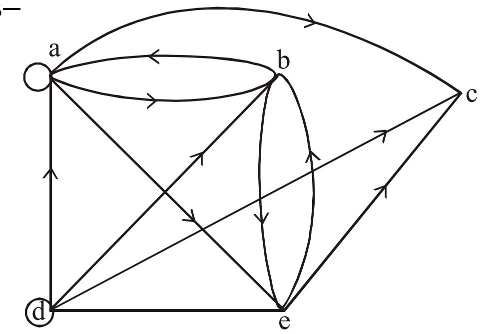
In a directed graph which of the following are paths–

(i) a, b, e, d; (ii) a, e, c, d, b;

(iii) b, a, c, b, a, a, b; (iv) d, c;

(iv) e, b, a; (v) e, b, a, b, a, b, e ?.

find their lengths. which of them are circuits.



Solution–

(i) Here each of (a, b), (b, e) and (e, d) is an edge so a, b, e, d is a path length three.

(ii) a, e, c, d, b is not a path because (c, d) is not an edge.

(iii) b, a, c, b, a, a, b is path of length six because

(b, a), (a, c), (e, b), (b, a), (a, a) and (a, b) are all edges.

(iv) (d, c) is an edge and so d, c is a path of length one .

(v) e, b, a is also a path and of length two because

(c, b) and (b, a) are edges. (vi) e, b, a, b, a, b, e is a path length six because (e, b), (b, a), (a, b), (b,a), (a,b) and (b, e) are edges.

The two paths b, a, c, b, a, a, b and e, b, a, b, a, b, e are circuits since they begin and end at the same vertex. However, paths a, b, e, d; c, b, a and d, e are not circuits.

Paths as relation–

Path can be applied as relations. we say that there is a path from a to b in R if these is a sequence of elements a, x_1 x_2 .... x_{n-1}, b with (a, x_1) $\in$R, (x_1, x_2) $\in$R, ... (x_{n-1}, b) $\in$R

2.20 Theorem

Let R be a relation on a set A. There is a path of length n (n $\in$N) from a to b if and only if (a, b) $\in R^n$.

Proof.

The proof is based on induction method.

When n = 1, means there is a path from a to b of length are iff (a, b) $\in$R. But this true from definition. Hence the statement is true for n = 1.

Now let the statement is true for positive integer n. (assume). There is a path of length (n+1) from a to b if there is an element c $\in$A such that there is a path of length one from a to c, so (a, c) $\in$R and a path of length n from c to b i.e. (c, b) $\in R^n$. Then by induction assumption. There is a path of length (n+1) from a to b iff there is an element c with $(a, c) \in R$ and $(c, b) \in R^n$. But there is such an element iff $(a, b) \in R^{n+1}$. Thus there is a path of length (n+1) from a to b iff $(a, b) \in R^{n+1}$. Hence the proved.

Definition 2.29-08

Connectivity Relation – Let R be a relation on a set A. The connectivity relation R^{-*} consists of the pairs (a, b) such that there is a path of length at least one from a to b in R.

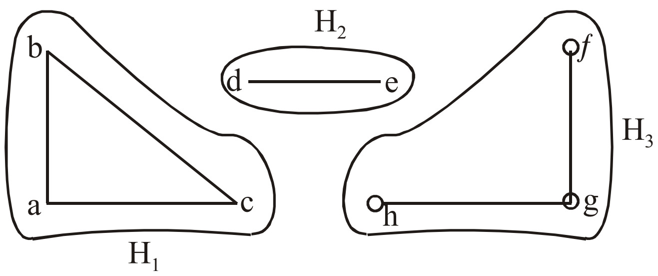
Since R^n consist of the pairs (a, b) such that there is a path of length n from a tob, it follows that R^* is the union of all the sets R^n. ie.

$$R^* = \bigcup_{n=1}^{\infty} R^n$$

We use this relation in many models.

3.29-09 Connected components –

A graph that is not connects is the union of two or more connected subgraphs, each pair of which has no vertex in common. These disjoints connected subgraphs are called the connected components of the graph.



Here H_1, H_2, H_3 are three subgraphs which form the connected components of H. (Which is union of H_1, H_2, H_3).

2.29-10 Def. Strongly connected –

A directed graph is called strongly connected if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

A directed graph can be strongly connected if there is a sequence of directed edges from any vertex in the graph to any other vertex.

A directed graph can fail to be strongly connected but still be in one piece.

2.29-11 Def weakly connected –

A directed graph is weakly connected if there is a path between every two vertices in the under lying undirected graph.

Hence a directed graph is weakly connected iff there is always a path between two vertices when the direction of the edges are disregarded.

Thus any strongly connected directed graph is also weakly connected.

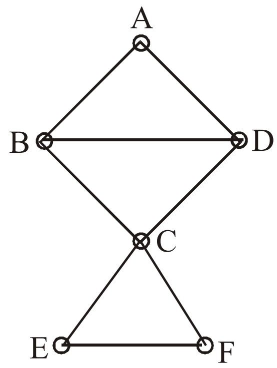
2.26 Example–1

In this graph A B C D E F C B D is a walk of length 7 which is neither a trail nor a path.

A B C E F C D is a trial but not a path.

A B C E F C D B A is a closed walk which is not a circuit.

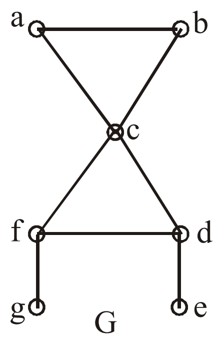
B C E F C D B is a circuit which is not a cycle. D E D B is a 3-cycle and hence an odd cycle. The closed walk C E F C B D C is not a cycle because the first and last vertex appears a third time.



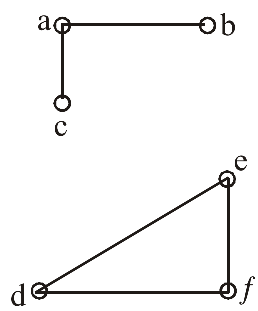
2.27 Example –

Here graph G is connected since for every pair distinct vertices there is a path between them.

graph G_1 is not connected because this is no path between a and d.



G



2.21 Theorem

There is a simple path between every pair of distinct vertices of a connected undirected graph.

Proof –

Let G = (V, E) be a connected undirected graph. Let u and v be two distinct vertices of G. As G is connected so at least one path between u and v exists. Let $x_0, x_1, \ldots\ldots\ldots x_n$ (n_0 = u and x_n= v) be the vertex sequence of a path of least length. We claim that this path is simple if not, let $x_i = x_j$ $(0 \le i < j)$. This means there is a path from u to v of shorter length with vertex sequence $x_0, x_1 \ldots\ldots\ x_{i-1}\ x_i \ldots\ldots\ x_n$ obtained by deleting the edges corresponding to the vertex sequence $x_i, \ldots\ldots\ldots, x_{j-1}$

2.29-12 Def. Null graph –

A graph G of order n and size 0 is called null graph (disconnected graph), it is denoted by N_n. clearly every vertex of null graph is an isolated vertex.

e.g. $N_4 = ::$

3.29-14

2.29.13 Def. Finite and infinite graph –

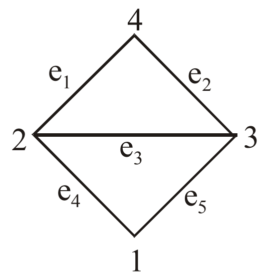
When a graph has only finite number of vertices and edges, then the graph is called finite. otherwise infinite.

2.29-14

Def. Trivial graph – It is a finite graph having a vertex and no edge. In other words, it is a graph with a single point.

2.29-15

Def. Labelled graph – A graph G in which edge are labelled with name or data is called a labelled graph .



Example. 2.27

Find the size of a K-regular graph.

Solution :

consider a K-regular graph G. then deg (v) = K, $\forall\, V \in V(G)$.

$$\Rightarrow 2|\,E\,| = \sum_1^n \deg(v) \qquad = \sum_1^n K = K \times n.$$

$$\Rightarrow |\,E\,| = \frac{K \times n}{2}.$$

2.29-16 Multigraph –

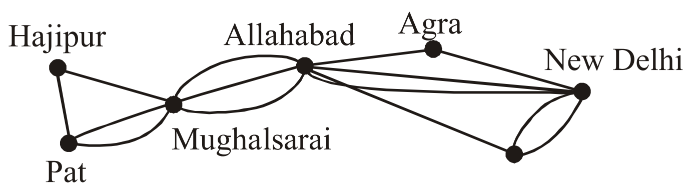
A multipgraph G consists of a set V of vertices, a set E of edges and a function f from E to $\{(u, v) : u, v \in V, u \neq v\}$.

Two edges e_1 and e_2 are called multiple or parallel

if $f(e_1) = f(e_2)$.

e.g. A road map.

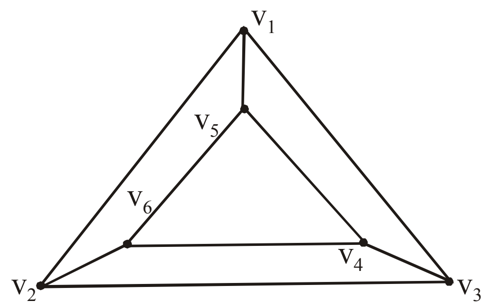
It should be noted that multiple edge in a multigraph are associated to the same pair of vertices.



2.28 Example. –

Example of a regular graph of degree 3

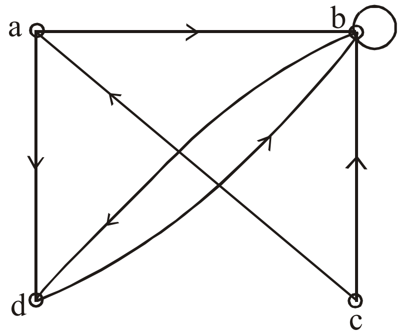
having even number of vertices.



2.29-18 Directed Graph A directed graph consists of a set V of vertices (or nodes) together with a set E of ordered pairs of elements of V called edges (or areas). If (a, b) is an edge then a is called initial vertex and b is called terminal vertex of this edge.

A loop is an edge of the from (a, a) form the vertex a to itself.

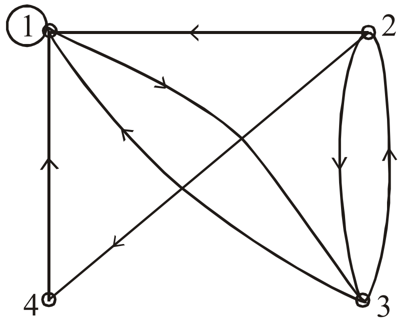
Example 2.29 : In the given figure :



This is directed graph with vertex a, b, c and d and edges (a, b), (a, d), (b, b), (b, d), (c, a), (c, b) and (d, b)

We use arrow sign pointing an edge (u, v) from u to v.

Example 3.291 :–



A directed graph of relation R = {(1, 1), (1, 3), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (4, 1)} on the set S = {1, 2, 3, 4} be presented as follows:

2.29-19 Def.

Directed multigraph – A directed multigraph G = (V, E) consists of a set V of vertices, a set E of edges and a function f from E to {(u, v) : u, v $\in$V}. The edges e_1 and e_2 are multiple edges if $f(e_1) = f(e_2)$.

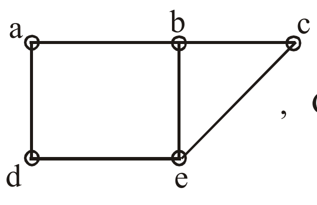
2.29-20 Def.

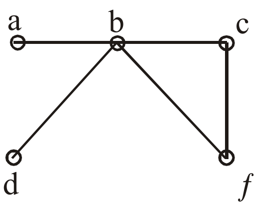
Union of two graph. – Union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of $G_1$and G_2 is directed by $G_1 \cup G_2$.

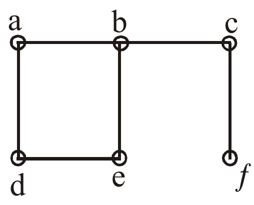
ie. $V(G_1 \cup G_2) = V(G_1) \cup V(G_2) = V(G)$ and

$E(G_1 \cup G_2) = E(G_1) \cup E(G_2) = E(G)$

Example : 2.292

$G_1 =$ 

, $G_2 =$ 

Then $G_1 \cup G_2 =$ 

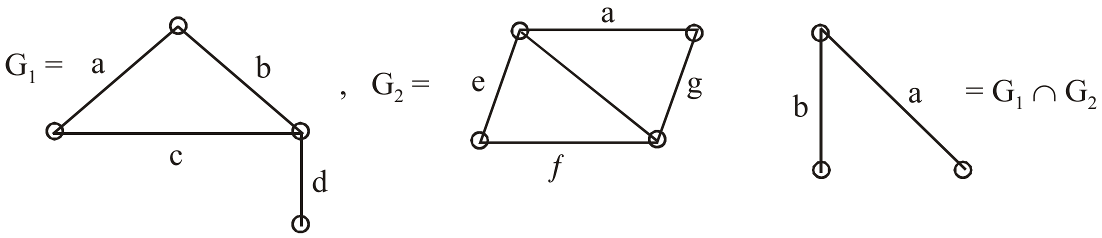
2.29-21 Def

Intersection of two graphs – Intersection of two simple graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with one vertex common is the simple graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. It is denoted by $G_1 \cap G_2$ ie.

$V(G_1 \cap G_2) = V(G_1) \cap V(G_2) = V(G)$ and

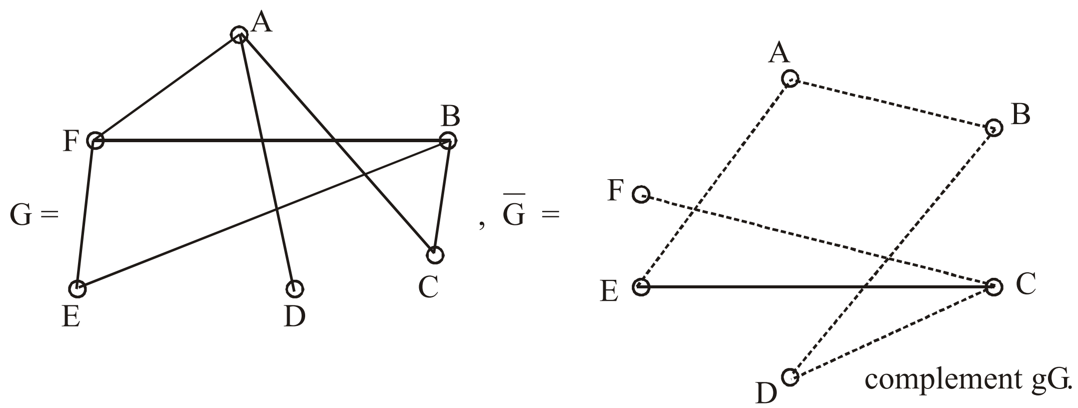
$E(G_1 \cap G_2) = E(G_1) \cap E(G_2) = E(G)$

For example 3.29.3



Def. 2.29-22 Complement of a graph – The complement of a graph G is a simple graph with the same vertex set as G in which two vertices are adjacent only when they are not adjacent in G. It is denoted by $\overline{G}$.

For example 3.29.4 :



2.21 NB. If we take complement of $\overline{G}$, we get back to G.

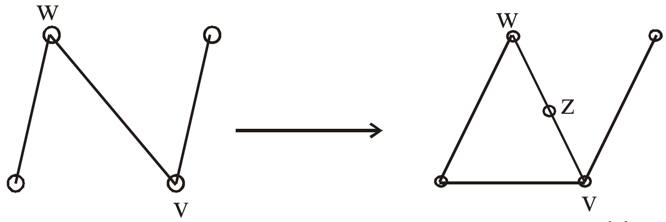
Def. 2.29-23

Homeomorphism : Two graphs G_1 and G_2 are called Homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivisiors ie. by introducing vertices of degree 2 in any edge of graph G_1.

If two graphs can be obtained from the same graph by inserting vertices into its edges then the two graphs are called homeomorphic.

2.29-23 Def. :

Inserting a vertex into an edge.– If G be a graph and e = vw be an edge of G then we can obtain a new graph by replacing e by two new edges vz and zw, Where z is a new vertex. This is called inserting a vertex into an edge.



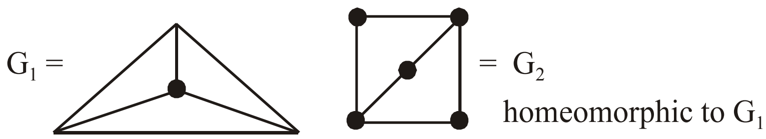
Homeomorphic graph

2.29-24 Def :

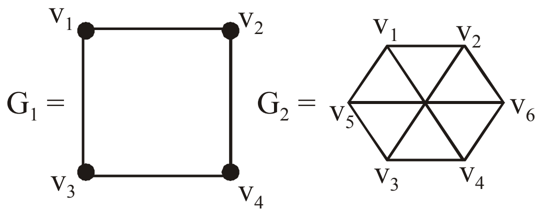
Contracting the edge e. – When a new graph is obtained from G by removing the edge e = vw and iductifying v and w in such a way that the resulting vertex is incident to all these edges (different from e) which were originally incident to V or to W, the process is called contracting the edge e. It is denoted by G/e.

2.29.5 Example of Homeomophic graph

(i)

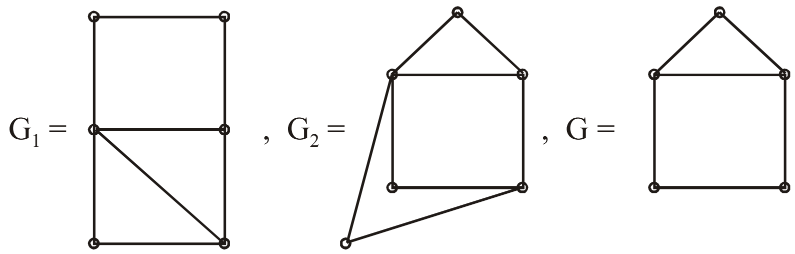


(ii)



Here G_1 and G_2 are Homeomorthic because G_2 has been obtained form G_1 by introducing vertices of degree 2 on edge $e_1 = V_1V_3$ and $e_2 = V_2V_4$.

Example 3.29.6 In the given graphs



Here G_1 and G_2 are Homeomorphic since both can be obtained from G by adding a vertex to one of its edges.

2.29-25

Def. : Automorphim : An automorphism of G is a one one, into mapping ϕ of V(G) onto itself with the property that ϕ (V) and ϕ (W) are adjacent if and only if V and W are one.

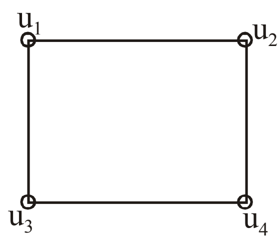
2.29-26

Isomorphism of Graphs. – Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there is one to one correspondence and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if f (a) and f (b) are adjacent in G_2 $\forall$ a, b $\in V_1$.

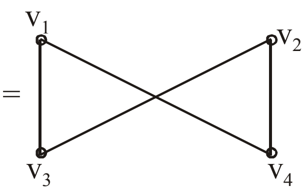
f is called isomorphic. (A Greek word where isos means equal and morphe means form). or essentially the same.

2.298 Example.

The graph $G = (V_1E) =$



and $H =$



Solution.

Define a function f as

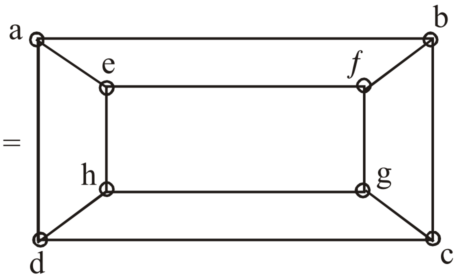
$f(u_1) = V_1, f(u_2) = V_4, f(u_3) = V_3, f(u_4) = V_2$

then f is a one to one correspondence between V and W. f-presence adjacenly also. Because the adjacent vertices in G and u_1 and u_2, u_1 and u_3, u_2 and u_4 and u_3 and u_4 and each of the pairs $f(u_1) = V_1$, $f(u_2) = V_4$ and $f(u_3) = V_3$, $f(u_2) = V_4$ and $f(u_4) = V_2$ and $f(u_3) = v_3$ and $f(u_4) = v_2$ are adjacent in H.

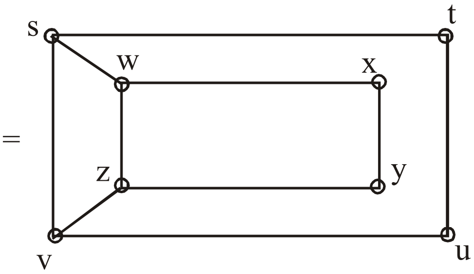
Example 3.298

Determine whether the graphs shown below are isomorphic.

G =



, H =



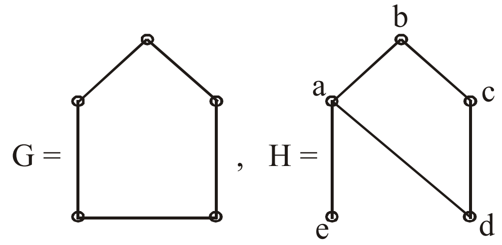
Solution :

Here G and H have eight vertices and 10-edges. Also they both have four vertices of degree 2 and four of degree 3. However, G and H are not isomorphic. Because, since deg(a)

= 2 in G, a must correspond to either t, u, x or y in H, since those are the vertices of degree two in H. However, each of those vertices in H is adjacent to another vertex of degree 2 in H, which is not true for a $\in$ G.

Example 3.299

Show that two graphs are not isomorphic,



Solution :

Here both G and H have five vertices and Six edges. But H has a vertex of degree one viz.e, whereas G has no vertices of degree one. So G and H are not isomorphic.

Some steps to establish isomorphism –

If G_1 is isomorphic to G_2 and $f: V(G_1) \to V(G_2)$ then

(1) $f(x) = f(y) \Rightarrow x = y$.

(2) Given $y \in V(G_2)\ \exists\ x \in V(G_1)$ s.t. $y = f(x)$

(3) $(x,y) \in E(G_1) \Leftrightarrow [f(x), f(y)] \in E(G_2)$

N.B. If $G_1 \cong G_2$ then

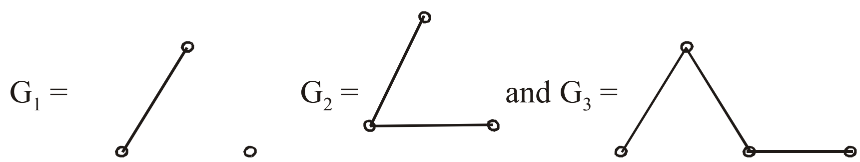
(i) G_1 and G_2 have same number of vertex and edges.

(ii) Same degree sequence.

2.29-27 Def.

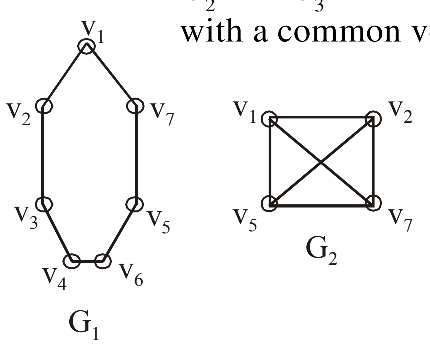
Self complementry graph – Let G be a graph and $\overline{G}$ be its complement then G is called self complementry if $G \cong \overline{G}$.

Example 3.29-01– Consider three graphs



Here G_1 is not isomorphic to G_2 or G_3 (as G_1 has only one edge)

G_2 and G_3 are isomorphic (Because G_2 and G_3 both have two edges incident with a common vertex.)

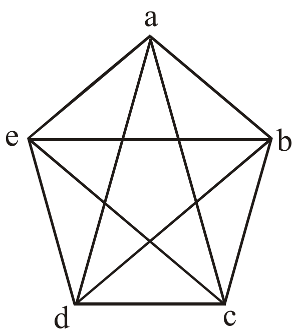


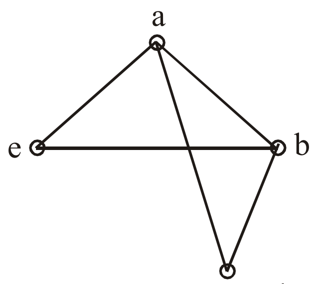
Here G_1 and G_2 are not isomorphic because they have different number of vertices.

2.30 Subgraph and Induced gaphs

2.30 Def. : Subgraph – If G = (V, E) be a graph then the graph H = (W, F) is called subgraph of G if $W \subseteq V$ and $F \subseteq E$.

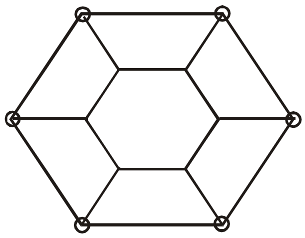
Example 2.30

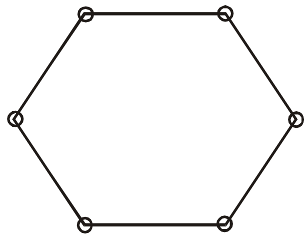




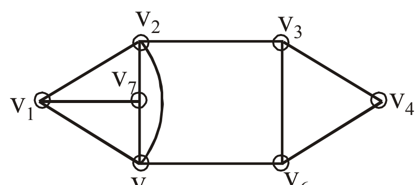
a subgraph

Example 2.31

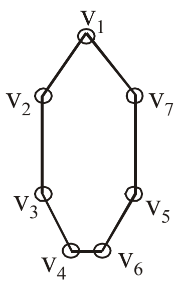

G_1


G_2 = a subgraph gh_1

Example 2.32



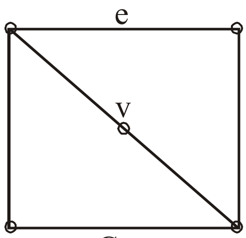
G



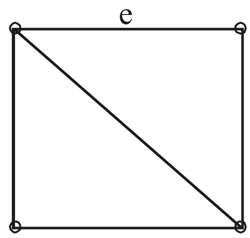
G_1

Here G_1 is a subgraph of G.

Example 2.33



G

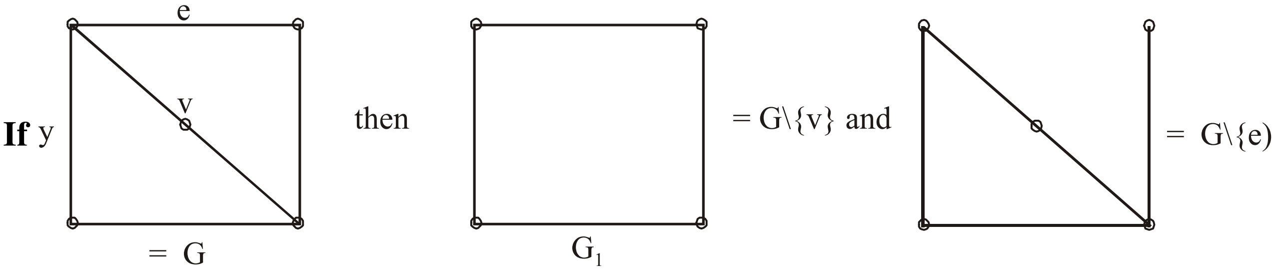


G_1

Here G_1 is not a subgraph of G.

In a graph theory, some times we need to delete an edge or a vertex from a graph G. We will denote by symbol G| [e] to emphasize that G has some number of vertex but whose edge is E| {e}. similar deletion of vertex V will be denoted by G|{V}

Example 3.34



N.B. 3.30 : In a subgraph of G. following is true –

(i) A graph G is subgraph of itself

(some concept used in the case of subset)

(ii) An isolated vertex and totally disconnected graph on V-vertices is a subgraph of a graph on n-vertices if $V \subseteq n$.

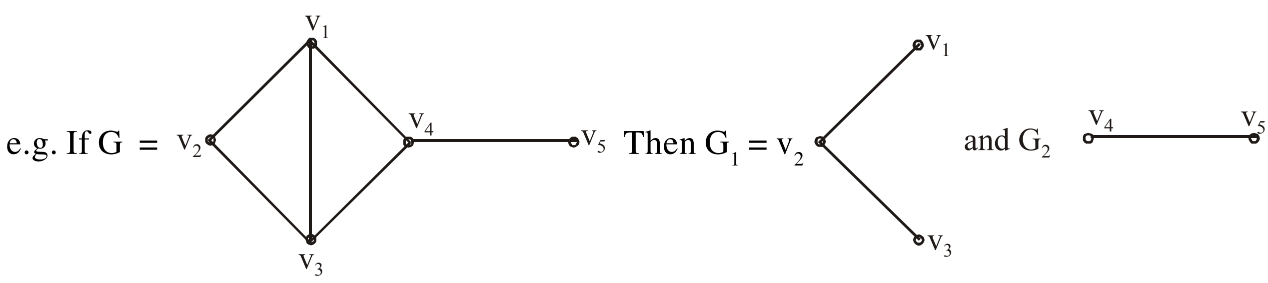
(iii) A subgraph H of G is called a proper subgraph of G if H is not isomorphic to G.

Def. 2.31 Disjoint subgraph –

It is of two types viz. (i) Edge- disjoint subgraph and (ii) vertex -disjoint subgraph.

2.32 Definition

(i) Edge- disjoint subgrah– (G-e) Let G be a graph and G_1 and G_2 be two subgrapghs of G. Then G_1 and G_2 are called edge disjoint if they have no edge in common symbolically, $E(G_1) \bigcap E(G_2) = \varphi$



are edge disjoint subgraph of G.

(ii) Vertex-disjoint subgraphs (G-V)– Let G be a graph and G_1, G_2 be two subgraphs of G. Then G_1 and G_2 are called vertex-disjoint subgraph of G if they have no vertex in common.

Symbolically, $V(G_1) \bigcap V(G_2) = \varphi$

2.35 Example Some as in (i) G_1 and G_2 are clearly vertex disjoint also.

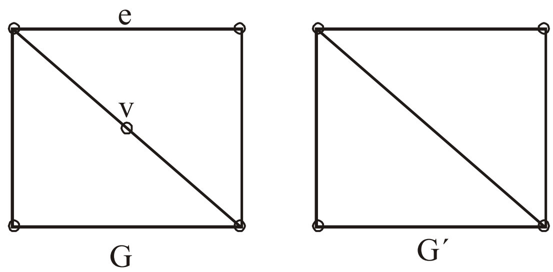
N.B. 2.31

(i) A vertex-disjoint Subgraph has also no edge in common ie. it is edge-disjoint also.

(ii) An edge-disjoint subgraph may have vertices in common hence an edge-disjoint subgraph is not necessarily vertex disjoint subgraph.

2.36 Example. Let G be a graph given below,

Is the graph G' a subgraph of G ?



Ans : No. G' is not a subgraph of G because the two vertices of degree 3 in G are not adjacent.

2.33 Def. Induced subgraph.

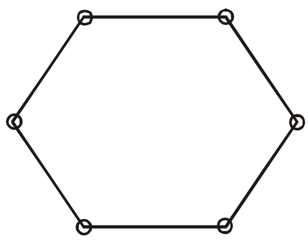
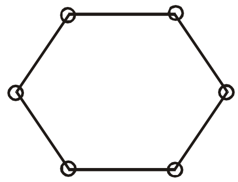
Let G = (V, E) be a simple graph and $W \subseteq V$. Then the subgraph induced by W is the graph (W,F) where the edge set F contains an edge in E if and only if both end points of this edge are in W. or,

In short, an induced graph of G is a subgraph which is induced by some subset W of V (G)

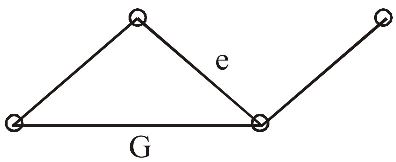
Def 2.34 Spanning subgraph.

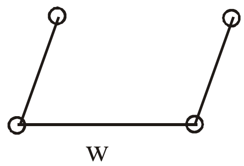
Let G = (V, E) be a graph then any subgraph W of G a called spanning subgraph if W contains all the vertices of G. ie. V(W) = V(G)..

2.37 Example (i) :

If G =  then w =  is a propt squaring subgraph of G.

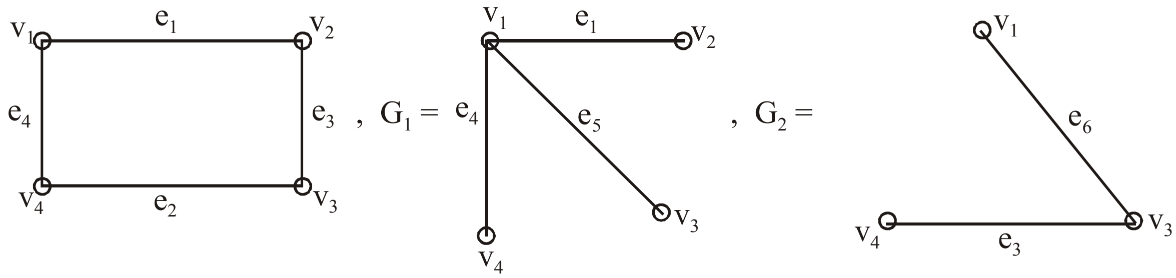
2.38 Example (ii) :





Def. 2.34 Complement of a subgraph

Let G = (V, E) be a simple graph. Let $G_1 = (V, E_1)$ and $G_2 = (V_2, E_2)$ be two subgraph of G. Then complement of subgraph G_1 is G_2 if $E_2 = E - E_1$ and V_2 contains only those vertices with which the edges in E_2 are incident.



Here G_1 and G_2 are subgraphs of G. Also G_2 is complement of G_1.

It is to be seen that some isolated vertices in G not induced in G_1, will not be included in G_2 either.

2.30 Theorem.

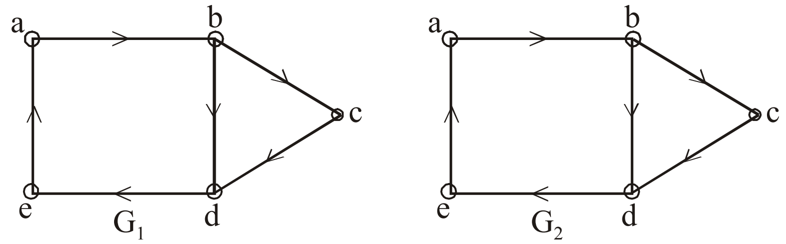
A pseudograph is Eulerian iff it is connected and every vertex is even.

Proof.

If part : Let the pseudograph G be Eulerian then obviosly, it is connected with each vertex even.

Only if part : Assume that G is connected with all vertices of even degree. To prove that G is Eulerian So. Let V be vertex of G. (If there is any loop incident with V, follow there first one after the other without any repetition). Since we are assuming that G has at least two vertices and G is connected so these must be an edge VV_1 ($V_1 \neq V$) incident with V. If there are loops incident with V_1, follow there one after the other without repetition. Then since deg V_1 is even (>0) so there $\exists$ an edge $V_1V_2 \neq VV_1$. Thus we have a trial from V to V_2 which we continue, if possible . Each time we arive at a vertex not encountered before follow all the loops without repetition. But the degree of each vertex is even we can leave any vertex different form v on an edge not get covered. Since pseudographs are always finite so this process can be repeated only finitely. So we will finally return to V having traced a circuit C_1. Also it can be seen that any vertex in G is even (Since we entered encountered) and left on different edges each time it was now if every edge have been covered then G is an Eulerian circuit and hence the proof. If not, we delete from G all the edges of G and all the vertices of G which are left isolated (i.e. acquire degree) by this procedure. All vertices of the remaining graph G_1 are even (since both G and G have only even vertices) and of positive degree. Also G_1 and G have one vertex-u as common (since G is connected). Hence starting at u and proceeding in G_1 as we did in G, we constuct a circuit C in G_1 which returns to u we now combining C and C_1 by starting at V moving along G to u, then through c back to u, and then back to V on the remaning edge of C_1 we then obtain a circuit C_2 in G which contains more edge than C_1 If it contains all the edges of G, it is Eulerian otherwise, we repeat the process, obtaining a sequence of large and larger circuit. Since our pseudograph is finite, the process must stop soon where and it stop only when with a circuit though all edges and vertices, that is with an Eulerian circuit.

3.39 Example. Do the two graphs strongly connected ?



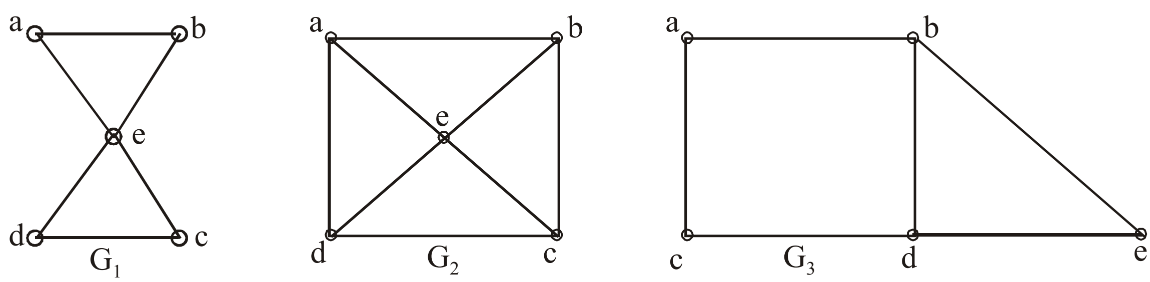
Answer –

G_1 is strongly connected because there is a path between any two vertices in this directed G_1 is weakly connected also.

The graph G_2 is not strongly connected as there is no directed path from a to b in this graph. But G_2 is weakly connected since there is a path between any two vertices in the underlying undirected graph of G_2.

Exapmle 3.391

Which of these graphs have an Eulter circuit and Eulerpath ?



Answer –

Here graph G_1 has on Eulter circuit, for example a, e, c, d, e, b, a. G_2 and G_3 have no Euler circuit. G_3 has an Eulerpath viz. a, c, d, e, b, d, a, b. But G_2 does not have an Eulerpath.

2.4 Degree of Vertex

Adjacent vertex –

2.40 Def. :

Two vertices u and v in an undirected graph G are called adjacent in G. if {u,v} is an edge of G. If e = {u,v} then the edge e is called incident with the vertices u and v. (end points of {u, v}.)

Def. 2.41

Degree of a vertex : It is the number of edges incident with it except that a loop at a vertex contributes twice to the degree of that vertex. deg(V) will be used to denote degree of V.

2.42. Def Isolated vertex –

It is a vertex of degree zero. clearly an isolated vertex can not be adjacent to any vertex.

Def. 2.43 Perdant vatex –

A vertex is called perdant iff it is of degree one. This means a perdent vertex is adjacent to exactly one other vertex.

Def. 2.44 In-degree and out-degree of a vertex.

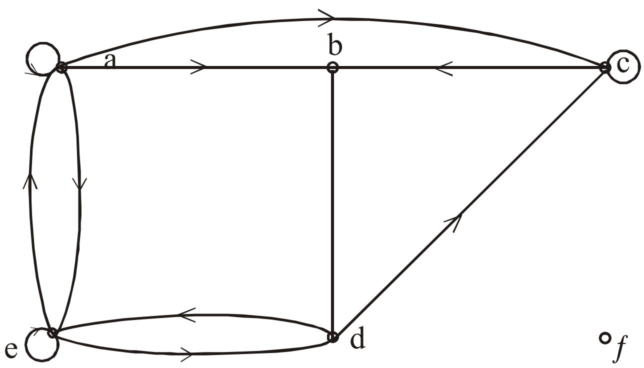
The in degree of a vertex V is the number of edges with V as their terminal vertex. and is denoted $\deg^-$ (v). The out-degree of a vertex V is the number of edges with V as their initial vertex and is denoted by $\deg^+(v)$.

2.40 N.B.

A loop at a vertex contributes 1 to both the in dgree and the out degree of this vertex.

Example 2.40

Find in-degree and out degree of each vertex in the graph G with directed edges with given figure



Solution –

Hence $\deg^-(a) = 2$, $\deg^-(b) = 2$, $\deg^-(e) = 2$, $\deg^-(c) = 3$, $\deg^-(d) = 2$, $\deg^-(e) = 3$ and $\deg^-(f) = 0$. The outdegree $\deg^+(a) = 4$, $\deg^+(b) = 1$, $\deg^+(c) = 2$, $\deg^+(d) = 2$, $\deg^+(e) = 3$, $\deg^-(f) = 0$

2.40 Proposition –

Prove that the sum of the degree of the vertices of a pseudograph is an even number equal to twice the number of edges ie

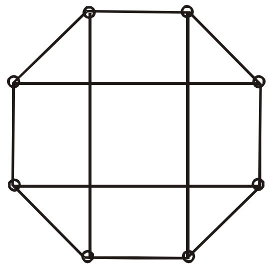
$\sum_{u \in v} \deg v = 2|E|$ where G (E,V) represents a pseudograph. (This is known as Euler's Proposition).

or (Handshaking theorem)

Proof.

Adding the degree of all the vertices involves counting one for each edge incident with each vertex. If the edge not a loop, it is incident with two different vertices and so gets counted twice, once at each vertex. If the edge is a loop, then a loop at vertex is also counted twice, in the degree of that vertex.

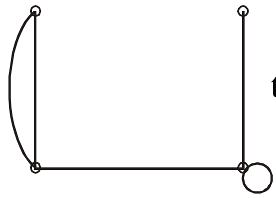
Example 2.41



Consider the figure It has eight vertices each of degree 3.

Now, $\sum \deg v = 8 \times 3 = 24$

If $2|\ E\ | = 24$ then E = 12, which is true.

Example. 2.42 In the figure,  the pseudograph

2.41 Theorem –

Prove that an undirected graph has even number of vertices of odd degree.

Proof :

Let G = (V, E) be an undirected graph. Let V_1 and V_2 be the set of vertices of even degree and vertices of odd degree respectively. Then.

We know that $2\,|\,E\,| = \sum_{v \in V} \deg(v)$

$$= \sum_{v \in V_1} \underset{(even)}{\deg(v)} + \sum_{v \in V_2} \underset{(odd)}{\deg(v)}$$

Since deg (v) is even for $v \in V_1$ so $\sum_{v \in V_1} \deg(v)$ is even. Also the sum of the two term on R.H.S is even = $2|\ E\ |$.

So, the second term on RHS. ie. $\sum_{v \in V_2} \deg(v)$ must be even.

But Since all the terms in this sum are odd so there must be an even number of such terms.

Hence we conclude that there are an even number of vertices of odd degree.

If (u,v) is an edge of the graph G with directed edges u is called adjacent to V and V is called adjacent to u. The vertex u is called initial vertex and V is called terminal vertex of (u,v). In the case of a loop the initial and terminal vertices are same.

Since the edges in graphs with directed edges are orderd pairs, the definition of the degree of a vertex can be refined to reflect the number of edges with this vertex as the initial vertex and as the terminal vertex.

2.43 Example

How many edges are there in a graph with ten vertices each of degree six.

Solution.–

As given , G has 10- vertex each of degree 6 so. the sum of the degrees of the vertices 6.10 = 60. Then using formula $2|E| = 60 \Rightarrow |E| = 30$

2.42 Theorem –

Let G = (V, E) be a graph with directed edges .

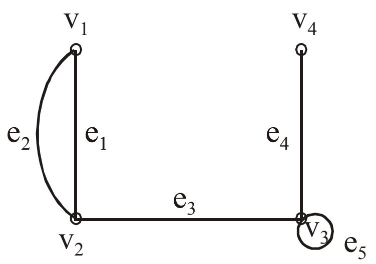
Then. $\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|$

Proof :

Proof not required.

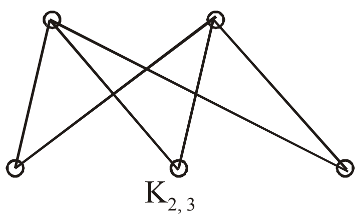
Def. : Degree sequence of G : Let G be a graph (or pseudograph) and $d_1, d_2, \ldots\ldots\ldots\ldots d_n$ be the degree of vertices of G, ordered so that $d_1 \geq d_2 \geq \ldots\ldots\ldots\ldots \geq d_n$. Then $d_1, d_2 \ldots\ldots\ldots\ldots d_n$ is called degree sequence of G.

Example 2.44



The degree sequence of this pseudograph is 4, 3, 2, 1

Example 2.45



The degree sequence of this graph is

3, 3, 2, 2, 2.

Example 2.46

How many edges does k_3, 6 contain ?

Ans – K_3, 6 have six-vertices of degree 3 and 3-vertices of degree 6. So,

$$\sum_{v \in V} \deg(v) = 6 \times (3) + 3 \times (6) = 36$$

$$\Rightarrow \ 2|E| = 36$$

$\Rightarrow \ |E| = 18$ So k_3, 6 have 18 edges.

Example 2.47

Can there exist a graph where degree sequence is 5, 4, 4, 3, 2, 1 ?

Ans – Here the sum of degree is 5 + 4 + 4 + 3 + 2 + 1 = 19 (odd)

Not an even number so there can not be degree sequence as given.

Example 2.48

Let G be a non-directed graph with 12 edges. If G has six vertices each of degree 3 and the remaning having degree 3 ; Find the minimum number of vertices of G.

Solution –

As given, number of edges in a graph = 12

So, $\sum \deg(V) = 2 \times 12 = 24$ (where V denotes a vertex).

$\because \quad \deg(V) = 2|E|$

Also let x = number of vertices each of whose degree is less then 3.

Then $\sum \deg(V_1) < 6 \times 3 + 3x$

$\Rightarrow \quad 24 < 18 + 3x \qquad \Rightarrow \quad 3x > 6$

$\Rightarrow \quad x > 2 \qquad$ hence x = 3.

So, minimum number of vertices of G = 3 + 6 = 9

Def. 2.45 (Complete graph) –

A graph in which every two vertices are adjacent is called a complete graph . Denote by K_p the complete graph with p-vertices and $\frac{1}{2}(p-1)p$. edges.

Cp - denote circuit graph of order p.

Np-denote null graph of order p (with p-vertices and no edges).

Def. 3.46 (Bipartite graph) –

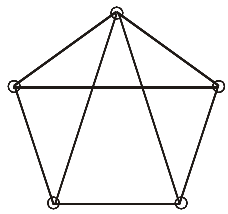
A graph whose vertex set can be partitioned into two sets (Called partite sets) in such a way that each edge joins a vertex of the first set to vertex of the second set.

N.B. A ϕ Bipartite graph can contains no triangles

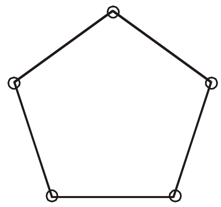
2.42

Example 2.49 –

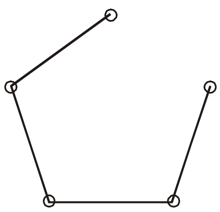
Ks-graph is shown by



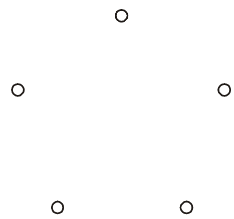
C_5-graph is shown by



P_5-graph is shown by



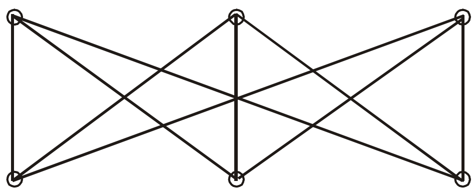
N_5-graph is shown



Complete Bipertite graph. –

Def. 2.46

In such a graph every vertex in the first set is adjacent to every vertex in the second set.

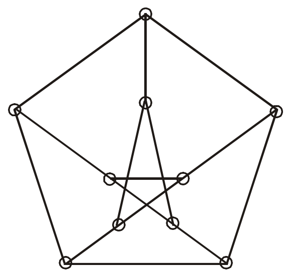


Kr,s denotes complete bipartite graph whose two partite sets contain r and s vertices, respectively.

2.47 (Petersen Graph) –

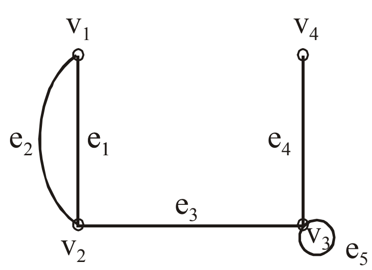
It is complement of the line graph of K_5.

E.g. K_1s is called star graph



Def. 2.47 Pseudograph. –

A pseudograph is like a graph but it may contain loops and /or multiple edges.



Theorem. 2.42 –

A simple complete graph with n-vertices has n(n – 1) edges.

Proof.

Since the graph G has n-vertices so any vertex can be selected in $^nC_1$ ways. Since the graph is complete so each vertex is connected to all other vertices. So a particular vertex V_1 can be connected to remaining (n – 1) vertices in (n – 1) ways. Since number of vertices is n So. the number of edges will be n(n –1).

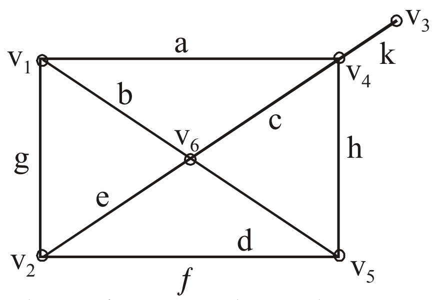
2.44 N.B.

If the graph contains n-loops then the number of edges will be = [n + n(n –1)] = n^2.

2.5 Connectivity Graph

Definition 2.50

Cut Set. – A cut-set in a connected graph G is a set of edges whose removal from G leaves G dis connected.



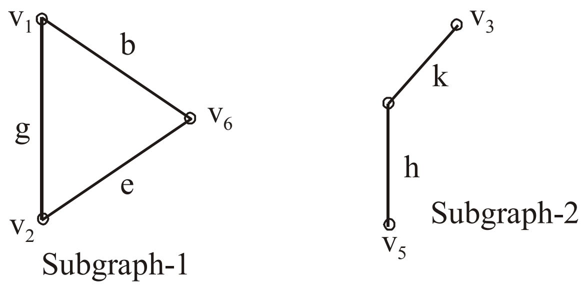
Example. 2.50

Let G be a connected graph having edges set

{a, b, c, d, e, f, g, h, k}.

If we remove edges a, c, d, f, G will be disconnected and cuts in two subgraph.

Hence the set g edges [a, c, d, f] is a cut set.



Definition 2.51

Edge connectivity of a graph : It is the minimum number of edges whose removal disconnects a connected graph G is called edge connectivity of G.

Definition 2.52

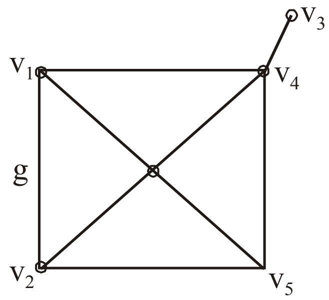
Minimally connected – A graph G is minimally connected if every edge of G is a cut set of G.

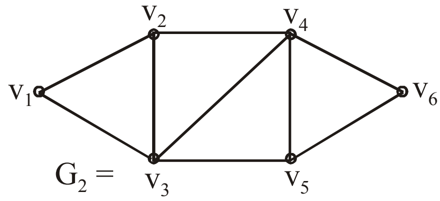
Definition 2.53

Vertex connectivity of a graph G : It is the minimum number of vertices whose removal result is a disconnected graph G.

Example 2.51

For the graphs G_1=





(1) edge connectivity of Graphs G_1 and G_2 is one and two respectively.

(2) Vertex connectivity of G_1 and G_2 are also one and two respectively.

(3) The edge connectivity of a tree is one and vertex connectivity of the tree is also one.

(4) Vertex connectivity of a disconnected graph is zero and of a graph having a bridge is always one.

Vertex connectivity is meaningful for those graphs only which have three or more vertices and are not complete.

Def. 2.54

Seprability of a graph – A connected graph is said to be seprable if its vertex connectivity is one. All other connected graphs are called non-seprable. In a seprable graph a vertex whose removal disconnects the path is called a cut-vertex, a cut- node or an articulation point.

For example, V_4 is a cut-vertex in $G_{1.}$

Theorem 2.50

A vertex V in a connected graph is a cut vertex if and only if there exists two vertices x and y in G such that every path between x and y passes through V.

Proof :

If part – As given a connected graph G has a cut vertex V so (G – V) is a diconnected graph. We select x and y in two different components of (G - V). Then three exists no path from *x* to *y* in (G – V). Also, G is connected so $\exists$ a path P from *x* to *y* in G. If the path P does not contain the vertex V, then removal of V from G will not disconnect the vertices *x* and *y* (Which is a contradiction) because x and y lies in two different components of (G – V).

Only if part – Let every path from *x* to *y* contains the vertex V. then removal of V from G disconnects *x* and *y*. Hence *x* and *y* lies in different components of G which implies that (G – V) is disconnected graph. So V is a cut vertex of G.

Theorem 2.51

The edge connectivity of a graph G is always greater than the vertex connectivity of G.

Proof :

Let C be the edge connectivity of a graph G. Then $\exists$ a cut-set S in G with C edges. Let V_1 and V_2 be the partitions of the vertex set of G with respect to S. Then the edges in S are the edges of G, V_1 and V_2. If no two edges in S have the same end vertex in the set V_1 or V_2. Then removal of all the end vertices of the edges in S disconnects G, the number of vertices required to disconnect G is less than the number of edges in S.

Example 2.52

The vertex connectivity of a graph G is always less than or equal to the edge connectivity.

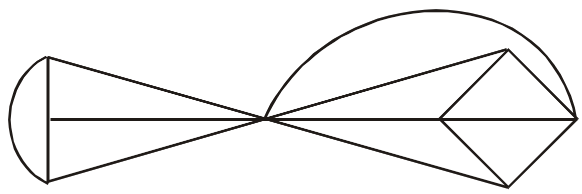
Proof :

For a disconnected or trivial graph G, the vertex connectivity = edge connectivity = 0. If G is connected by a bridge means single edge then edge connectivity = 1 and then vertex connectivity is also 1.

So let the edge connectivity ≥ 2. Then G has e_1 lines whose removal disconnects G. Hence the $(e_1 - 1)$ of those edges produce a graph with a bridge $e_1 = \{u, v\}$. For each of those $(e_1 - 1)$ edges select an incident point which is different from u or v. The removal of those vertices will also remove $(e - 1)$ edges and if the resulting graph is disconnected then vertex connectivity $\leq (e_1 - 1) <$ edge connectivity if edge $e = \{u, v\}$ is not a bridge and hence the removal of u and v will result in either a disconnected or a trivial graph. So vertex connectivity $\leq$ edge connectivity in all the case.

Example 2.53

For the given graph G, find both edge and vertex connectivity

G = 

Solution :

The minimum number of edges removal disconnecting the graph is 3 and so edge connectivity = 3.

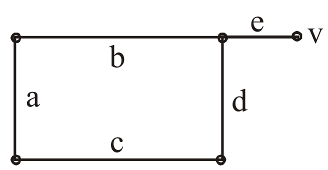
The minimum number of vertices requaired to disconnect the graph is 1. Hence vertex connectivity is also one.

Example 2.54

The edge connectivity of a graph is less than or equal to the degree of the vertex with the smallest vertex in G.

Proof :

Let V be the vertex with smallest degree. Then V can be seprated from G by removing at least d(V) edges. Hence edge connectivity $\leq$ d(u).



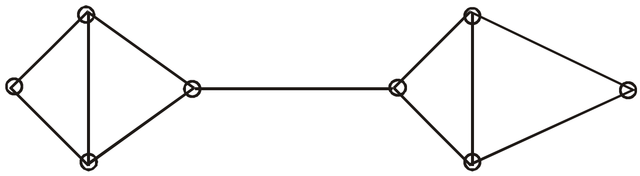
But degree of V is one. (minimum degree vertex). So if we disconnect e which gives the edge connectivity one. So edge connectivity is equal to the degree of the vertex with the smallest degree in G.

Theorem 2.51

The edge connectivity of a graph G cannot exceed the minimum degree of a vertex in G.

Proof :

As given G is an edge connectivity graph. Let V be a vertex of minimum degree in G. Now removal of all the edges incident with V disconnects the vertex V from G. The set of all edges incident with V in a cut set of G. But edge connectivity is the minimum cardinality of the cut sets of G, which implies that the edge connectivity is always less than the number of chord incident with the vertex. The chord connectivity need not lie equal to the minimum degree of a vertex of G . For example, in the given graph the edge connectivity is one but the minimum degree of a vertex of the graph is two.



3.50 Dual of a gaph

If G is a connected plane graph, then its dual graph G* is the graph obtained by the following procedure:

(i) Place a pont inside each face of G - these points corresponds to the vertices of G*.

(ii) for each edge e of G, draw a line joining the vertices in the two faces bounded by e-these lines corresponds to the edges of G*.

3.51 Important points on Dual of a graph –

(i) An edge forming a self loop in G yeild a Pendant edge in G*.

(ii) A Pendant edge in G yields a self-loop in G*.

(iii) Edges that are series in G produce parallel edges in G*.

(iv) Edges that are parallel in G produce series edge in G*.

(v) The number of vertices in dual of a graph is equal to number of regions in the planar of G .

(vi) We have the number of edges in G and G*.

(vii) Dual of a graph is the graph itself.

(viii) The nullity of G is the rank of G* and nullity of G* in the rank of G.

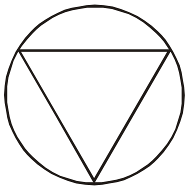
Thus is short dual of G* of a polygonal graph G is a polygonal graph each of whose vertices correspond to a face of G and each of whose vertices correspond to a vertex of G. Two vertices in G* are connected by an edge if the corresponding faces in G have a boundary edge in common.

2.55 Def.

Self dual graph – A planar graph G isomorphic to its own dual is called self dual graph.

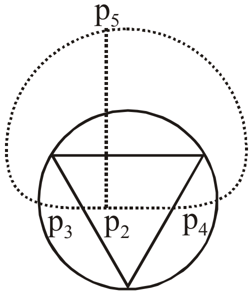
Example 2.55

Find dual of the graph G.



Solution :

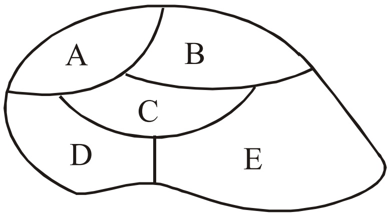
The dual of a graph depends on the embedding of the graph in the plane and the same graph may have different geometric dual for different embeddings.



Dual of graph G

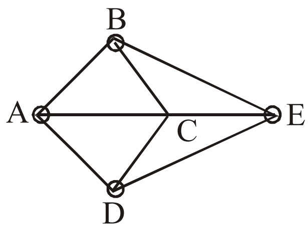
Example 2.56

Find dual of graph G.



Solution :

The dual of the above graph is

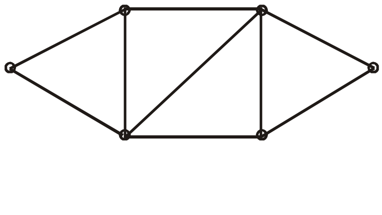


Example 2.57

Give example of a graph whose vertex connectivity is equal to the edge connectivity.

Solution :

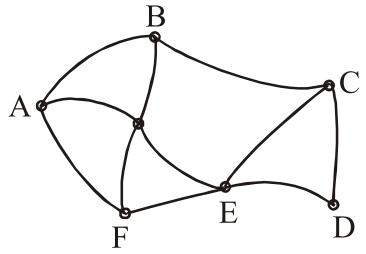
For the graph we can see that vertex connectivity = edge connectivity = 2.



2.6 Planar Graph

Definition 2.60

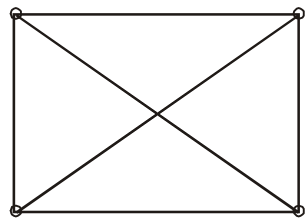
A graph which can be drawn in such a way that the edges have no intersection or common points except at the vertices is called a planar graph. The following graph is a planar graph : It is a road map showing connecting between various road junction.



It indicates that there are 7 junction, A to G, some of which are directly connected by roads. (A,G), (B,C), (F,E) etc. Hence a road map is a planar graph.

2.60 Example :

Is the given graph Figure plannar ?



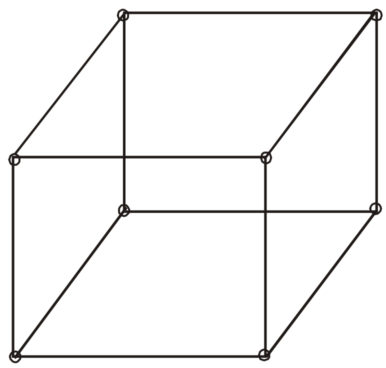
K_4-graphs

Solution –

Yes, it is a planar gaph because it can be drawn without crossing.

2.61 Example :

Is the graph



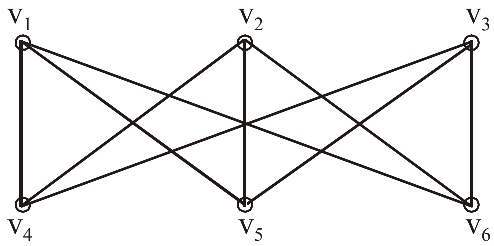
planar ?

Solution –

Yes, it is a planar graph because it can be drawn without any edges crossing.

2.62 Example :

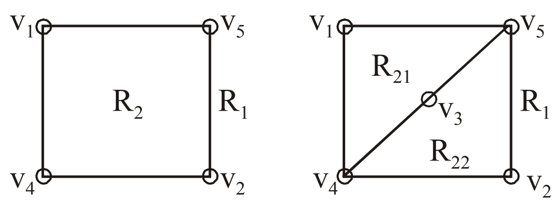
Is the graph Figure planar ?



K_{33} – graph

Solution –

No. it is not planar. In fact it is not possible to draw it is a plane with no edges crossing. Because, in any planar representation, the vertices V_1 and V_2 must be connected to both V_4 and V_5. These four edges form a closed curve which splits the plane into two regions R_1 and R_2 as shown below :–



The vertex V_3 is in either R_1 or R_2. If V_3 is in R_2 than the inside of the closed curve, the edges between V_3 and V_4 and between V_3 and V_5 seprate R_2 into two subregion R_{21} and R_{22}

Also, there is no way to place the final vertex V_6 without forcing as crossing. Because if V_6 is in R_1, then the edges between V_6 and V_3 cannot be drawn without a crossing. If V_6 is in R_{21}, then the edges between V_2 and BV_6 cannot be drawn without a crossing. If V_6 is in R_{22}, then the edge between V_1 and V_6 cannot be drawn without a crossing. Same will hold if V_3 is in R_1. thus the given graph is not planar.

Planarity of graph plays an important role in the design of electronic circuit. We can model a circuit with a graph by representing components of the circuit by vertices and connection between them by edges. If the graph of the circuit is a planar graph then it can be printed on a single board with no connections crossing. If the graph is not planar then some more expensive options can be found.

2.60 Euler's Formula–

A planar graph splits the plane into regions, including an unbounded region. Euler showed that all planar graphs splits the plane into the same number of regions. For this he found a relationship among the number of regions, the number of vertices, and the number of edges of a planar graph.

Statement :

Let G be a connected planar simple graph with e edges and V-vertices. Let r be the number of regions in a planar representation of G. then $r = e - V + 2$

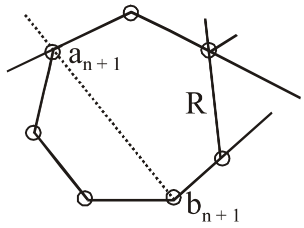
Proof :

Let G be a planar graph and G_1, G_2, ...$G_e = G$ be a sequence of subgraphs of G. Now pick one edge of G to obtain G_1. Similarly, obtain G_n from G_{n-1} by arbitrarily adding an edge that is incident with a vertex already G_{n-1} adding the other vertex incident with this edge if it is not already is G_{n-1} This is possible since G is connected. G is obtained after e edges are added. Let r_n, e_n and V_n represent the number of regions, edges and vertices of the planar representation of G_n induced by the planar representation of G, respectively

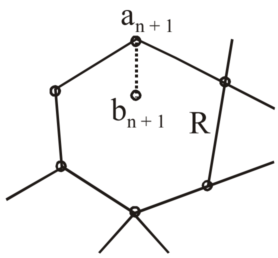
The proof is based on induction method. Clearly for G_1, $r_1 = e_1 - V_1 + 2$ is true for $e_1 = 1_1 V_1 = 2, r_1 = 1$. So assume that $r_n = e_n - V_n + 2$. Let $\{a_{n+1}, b_{n+1}\}$ be the edge that is added to G_n to obtain G_{n+1}. This is possible in two ways .... (i) in the first case both a_{n+1} and b_{n+1} are already in G_n. These two vertices must be on the boundary of a common region R, or else it would be impossible to add the edge $\{a_{n+1}, b_{n+1}\}$ to G_n without two edges crossing (G_{n+1} is planar). The addition of this new edge splits R into two region. Consequently in this case $r_{n+1} = r_n + 1$, $e_{n+1} = e_n + 1$ and $V_{n+1} = V_n$ Thus, each side of the formula relating the number of regions, edges and vertices increased by exactly one, So this formula is still true. Hence $r_{n+1} = e_{n+1} - V_{n+1} + 2$

(ii) **in the second case** one of the two vertices of the new edge is not already in G_n. Suppose a_{n+1} is not in G_n but that b_{n+1} is not. Adding this new edge does not produce any new regions, since b_{n+1} must be in a region that has a_{n+1} on its boundary. Consequently, $r_{n+1} = r_n$. More over, $e_{n+1} = e_n+1$ and $V_{n+1} = V_n+1$. Each side of the formula relating the number of regions, edges and vertices remains the same so the formula is still true. In other words, $r_n+1 = e_{n+1}- V_{n+1} +2$.

For case (i) figure is



For case (ii)



This proves the induction Law. Hence $V_n = e_n - V_n+2$, for all n. Since the original graph is the graph G_e, obtained after e edges have been added. So the theorem is established.

Example 2.63

Let a connected planar simple graph has 20 vertices, each of degree 3. Into how many region does a representation of this planar graph split the plane ?

Solution –

This graph has 20 vertices wach of degree 3, so that V = 20. Since the sum of the degree of vertices, $3V = 3 \times 20 = 60$, is equal to like the number of edges 2e so 2e = 60 so e = 30. So using Eulers, theorem.

$r= e - v + 2 = 30 - 20 + 2 = 12$. So number of regions = 12.

2.61 Euler's formula – (An alternative approch)

To prove that $V - E + R = 2$. Where symbols have usual meaning and graph G is connected.

Proof :

The proof is based on induction on E.

Let E = 0 then V = R = 1 (as G is connected) and hence the formula hold.

Assume that the formula holds for connected planar graph with (E – 1) edges, ($E \geq 1$). Now two case arise :– (i) When G contains no circuit. Then G is a tree so E = V – 1 (We know that a connected graph with n- vertices is a tree iff it has (n – 1) edges). Also R = 1 So $V - E + R = V - (V - 1) + 1= 2$ (ii) When G contains a circuit C. Then let e be any edge of e and consider the subgraph G/{e} which is a plane graph and still connected. The circuit C determines a region of G which disappears in G/{e}. So G/{e} contains (R – 1) regions, all V vertices of G, and (E – 1) edges. So by induction hypothesis.

$V - (E - 1) + (R - 1) =$ ie $V - E + R = 2$

2.62 Theorem :

If G is a connected planar simple graph with e edges and V-vertices ($V \geq 3$), then $e \leq 3V - 6$.

Proof :

A connected planar simple graph drawn in the plane divides the plane into regions (say r). The degree of each region is at least three (since we are discussing simple graphs so no multiple edges that could produce regions of degree two, or loops that could produce regions of degree one are permitted). In particular note that the degree of the unbounded region is at least since there are at least three vertices in the graph.

Also the sum of the degree of the regions is exactly twice the number of edges in the graph because each edge occurs on the boundary of the region exactly twice (either in two different regions, or twice in the same region). Since each region has degree greater than or equal to three, it follows that

$$2e = \sum_{\text{all region } R} \deg(R) \geq 3r \qquad \text{Hence, } (2/3)\, e \geq r$$

$$\Rightarrow \quad e - V + 2 \leq \frac{2}{3}e \text{ (using Euler's formula)}$$

$$\Rightarrow \quad \frac{e}{3} \leq V - 2$$

$$\Rightarrow \quad e \leq 3V - 6$$

2.63 Theorem :

If G is connected planar simple graph, then G has a vertex of degree not exceeding five.

Proof :

The result hold good if G has one or two vertices. If G has at least three vertices ie. $V \geq 3$ then since $e \leq 3V - 6$ so $2e \leq 6V - 12$. If the degree of every vertex were at least six, then since $2e = \sum_{\vartheta \in V} \deg(\vartheta)$, (Handshaking theorem), we have $2e \geq 6V$ (a contradiction). Hence there must be a vertex with degree no greater than five.

2.64 Theorem (Kuratowski's theorem).

A graph G is planar if and only if either

(i) G has no subgraph homeomorphic to K_5 or K_3, 3.

(ii) G has no subgraph contractible to K_5 of K_3, 3

Proof :

The proof is beyond the scope–

2.64 Example

Is the graph K_5 coplanar ?

Solution :

We know that the graph K_5 has four vertices and ten edges.

i.e. V = 5, e = 10.

then the condition of planarity

$e \leq 3V - 6$

ie. $10 \leq 3 \times 5 - 9$ is not satisfied.

Hence K_5 is not a planar graph.

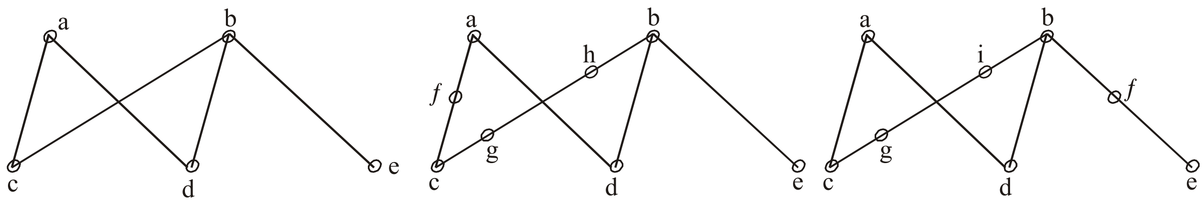
2.60 Kuratowski's Two Graphs –

We have seen that $K_{3},_{3}$ and K_5 are not planar So, a graph is not planar if it contains either of these two graphs as a subgraph. Suprisingly, all non coplanar graphs must contain a subgraph that be obtained from $K_{3},_{3}$ or K_5 using certain permitted operations. If a graph is planar then all other graphs obtained by removing an edge {u, v} and adding a new vertex w together with edges {u, w} and {w, v}. Such an operation is called homeomorphic subdivision.

2.61 Def.

The graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are all Homeomorphic if they can be obtained from the same graph by a sequence of elementary subdivision.

e.g. The graphs



are all homeomorphic graphs.

2.65 Theorem :

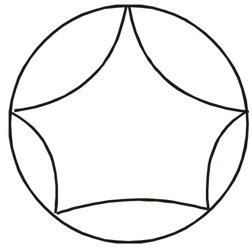
A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5

Proof :

The necessary part is quite obvious because we know that a graph contains a subgraph homeomorphic to $K_{3,3}$ or K_5 is non coplanar. However the converse part is complicated and hence beyond the scope.

2.65 Example :

Count the number of vertices, number of edges and number of regions of the following:



Solution :

Here V = 5, E = 10, Then using Euler's formula.

We have $V - E + R = 2 \Rightarrow R = 2 + R - V$

$= 2 + 10 - 5 = 7$

So number of regions = 7

2.66 Example :

A complete graph of n vertices is planar if $n \leq 4$, prove.

Proof :

For n = 1 to 4, the graph is always planar if connected. So when n = 5, and taking the graph complete, we have on using Euler's formula.

$$V - E + R = 2 \Rightarrow R = E - V + 2$$
$$= 10 - 5 + 2$$
$$= 7$$

But for a planar graph we know that $R \leq \dfrac{2E}{3}$

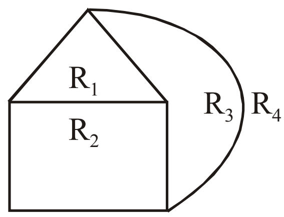
ie. $7 \leq \dfrac{2E}{3}$ ie, $21 \leq 2 \times 10$ (absurd).

and hence the complete graph with 5–vertices is non-planar.

Similarly, we can see that for n = 6, 7. the graph is non-planar. so $n \leq 4$.

2.67 Example :

Find degree of each region in the graph



Solution –

Here degree $R_1 = 3$, degree $R_2 = 4$,

degree $R_3 = 3$, degree $R_4 = 4$

2.61 STEPS TO SHOW PLANARITY

(i) cheek first for each component of graph G.

(Since planarity of graph $\Leftrightarrow$ component is planar).

(ii) Since planarity graph $\Leftrightarrow$ blocks are planar So, chek for each blocks of G.

(iii) Replace all parallel edges by an edge. (If it is possible to write an edge without crossing will each other then it is also possible to write more edges between the same pair of vertices without crossing each other. So, parallel edges or self loops does not affect planarity).

(iv) Remove all self loops.

(v) Replace all edges in series by an edge (two edges are in series provided they have one and only one vertex of degree two in common. So, marging of two series edge does not change the planarity.)

2.66 Theorem :

Prove that $K_{3,3}$ is non coplanar.

Proof :

Kuratowski second graph ($K_{3,3}$) is a regular connected graph having vertices and nine edges.

i.e. E = 9, V = 6 and R = 7

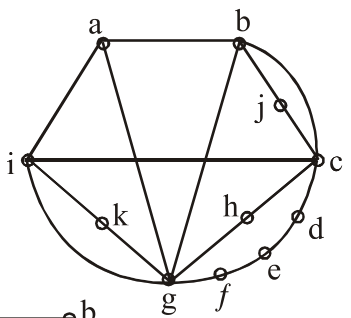
So, formula V – E + R = 2, for coplanarity $\Rightarrow$ 6 – 9 + 7 = 2

$\Rightarrow$ 4 = 2 (absurd).

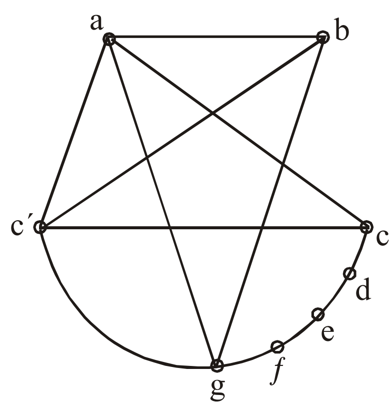
and hence $K_{3,3}$ is non- coplanar.

2.68 Example.

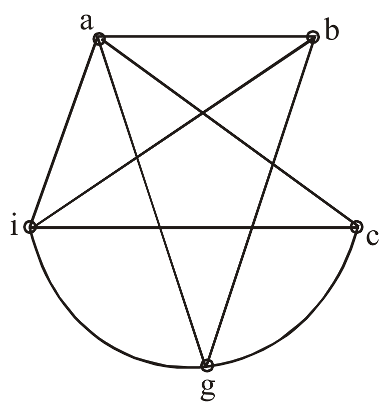
Is the graph given below is planar ?

G = 

Solution –

Consider a subgraph H = 

and another graph K_5 =

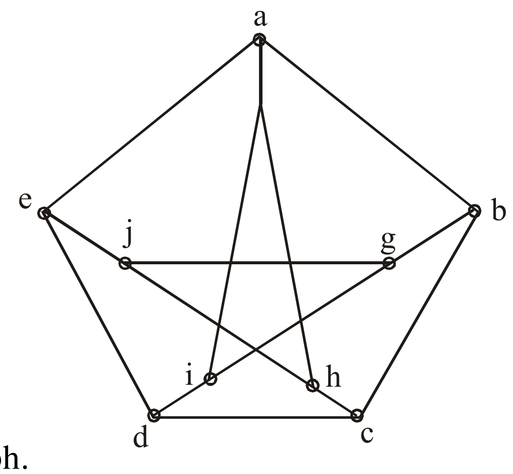


Then H is subgraph of G obtained by deleting h, j or and K and all edges incident with these vertices.

Also H is homeomorphic to K_5 since it can be obtained form K_5 by a sequence of elementary subdivision, adding the vertices d, e, f. Hence G is nonplanar.

2.69 Example.

Is the given graph G planar ?

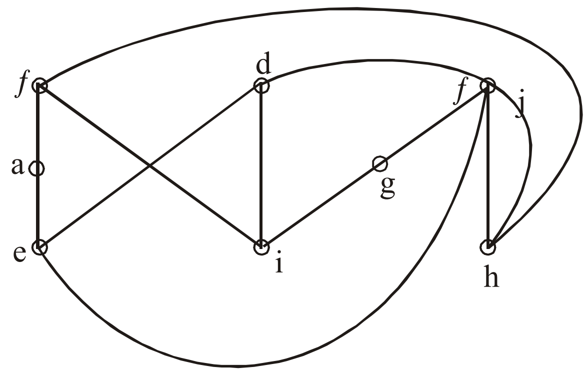


Solution –

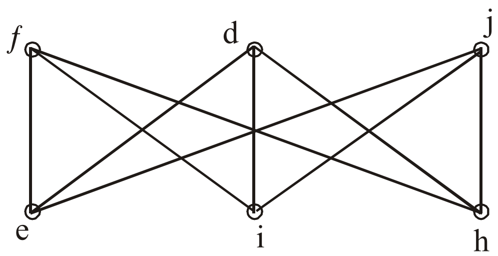
The given graph G is called Peterson graph.

Consider a subgraph H of G obtained by deleting b and the three edges that have b as an end point.

Then H =



clearly H is homeomorphic to $K_{3,3}$ =

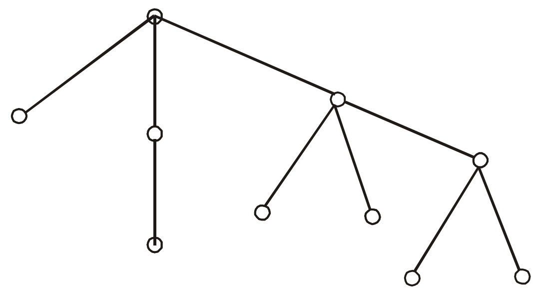


with vertex sets {f, d, j} and {e, i, h}, since it can be obtained by a sequence of elementary subdivision, deleting {d, h} and adding {e, h} and {e, d}, deleting {e, f} and adding {a, e} and {a, f}, and deleting {i,j} and adding {g, i} and {g, j}. Hence G is not planar (ie non planar).

2.7 Trees

A tree is a connected graph having no simple circuit. Arthur cayley actually used the trees in 1857 to count certain types of chemical compounds. It has got vast application in disciplines like computer since (in algorithms), games like chess and checkers, to model procedures in decision making etc. Family trees are graphs representing genealogical charts. Family trees use vertices to represent the member of a family and edges to represent parent child relationship.

In a connected graph it is possible to move from any node to any node (because they can be drawn in one piece) but in a tree it is not possible to move in a circular fashion throught any part of the graph.



Since a tree cannot have a simple circuit Fig a tree cannot contain multiple edges or loops. Hence a tree must be a simple graph.

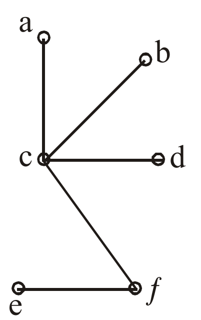
2.70 Def.

Directed tree – A directed tree is an a cyclic ghraph which has one node (called root) with in degree 0 and all ather nodes have in degree.

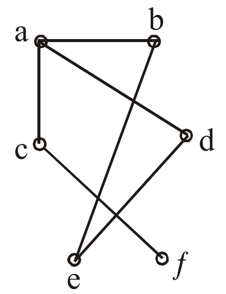
This means a directed tree must have at least one node.

2.70 Example.

Is the following graph a tree ?



(1)



(2)

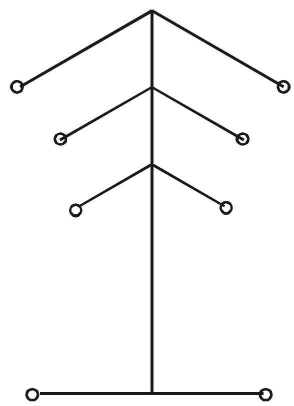
Solution. –

Here figure (1) is a tree since it is a connected graph. with no simple circuit.

But (2) is not a tree since e, b, a, d, e, is a simple circuit in the graph.

We have defined a tree as a connected graph contains no simple circuit and nof necessarily connected. Such graphs are called forest It has the property that each of their connected components is a tree.

The example of a forest is



2.70 Theorem

An undirected graph is a tree iff there is a unique simple path between any two of its vertices.

Proof :

Let T be a tree. This means T is a connected graph with no simple circuits. Consider two vertices x and y of T. Then there is a simple path between x and y ($\because$ T is connected) which is unique also otherwise, if there exists a second path, the path formed by combining the first path form x to y followed by the path from y to x (obtained by reversing the order of the second path from x to y) would form a circuit. This will mean that there is a simple circuit in T. Hence there is a unique simple path between any two vertices of a tree.

So, next assume that there is a unique simple path between any two vertices of a graph T. Then T is connected as there is a path between any two of its vertices. Also T can have no simple circuits. (Because, if possible let T had a simple that contained the vertices x and y. Then there would be two simple paths between x and y, since the simple circuit is made up of a simple path from x to y and a second simple path from y to x). Hence a graph with unique simple path between any two vertices is a tree.

2.71 Def. Rooted tree –

A rooted tree is a tree in which one vertex has been disignated as the root and every edge is directed away from the root.

2.72 Def.

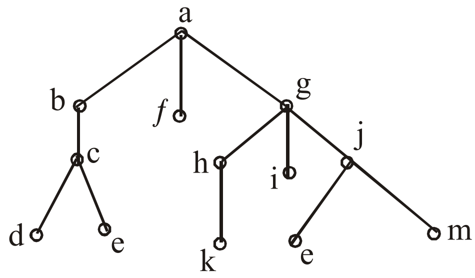
m-ary tree – A rooted is called m-aray tree if every internal vertex has no more then . Some important terminologies (resembling Botanical and genealogical origins).

Let T be a rooted tree and V be a vertex in T other than the root. Then parent of V is the unique vertex u such that there is a directed edge from u to V. When u is the parent of V then V is called child of u. Vertices with the same parents ae called siblings

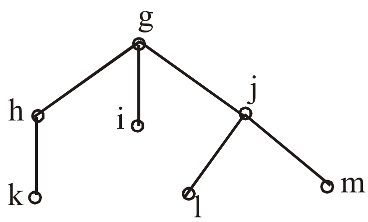
The ancestors of a vertex other than the root are the vertices in the path from the root to this vertex, excluding the vertex itself and including the root. The decendats of a vertex V are this vertices that have V as an ancestor. A vertex of a tree is called a leaf it has no children. Vertices that have children are called internal vertices. The root is an internal vertex unless it is the only vertex in the graph in which case it is a leaf.

A subtree with a (vertex a in a tree) as its root is the subgraph of the tree consisting of a and its decendants and all edges incident to these descendants.

2.71 Example :–



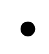
In the given figure find the parent of e, the children of g the sibling of h, all ancestors of e, all decendents of b, all internal vertices, and all leaves. What is the subtree rooted at g.



Ans – In the rooted tree T, b is parent of e., h, i, j are children of g. i and j sibling of h e, b and a are ancestors of e. c, d, e are decendants of b. The internal vertices are a, b, c, g, h and j. d, e, f, c, k land in are leaaves. The subtree rooted at g is Fig

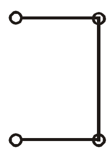
2.72 Some Example

We present trees with different number of vertices–

(i) With one vertex 

(ii) With two vertex 

(iii) with three vertices 

(iv) with four vertices 

Def. 2.72 Trivial tree –

A tree with only one vetex is called a trivial tree.

2.73 Def

Pendant vertex– A vertex of degree one is called a perdant vertex.

A tree with two or more vertices has at least two Perdant vertices.

Every tree has a perdant vertex (Because otherwise, every vertex will lie in a circuit of the graph) which is not possible as in a tree a graph must be a cyclic.

Def. 2.74 Distance between two vertices – Let G be a graph and u and v between vertices connected by a path then distance between them is the length of shortest path between u and v. It is devoted by d (u,v).

If there is no path connecting them then we say that d (u, v) = ∞

Def. 2.75

Eccentricity of v – It is denoted by e(v) and defined as $e(v) = \max^m \{d(u,v) : u \in V, u \neq v\}$, V is set of all vertices v.

Def. 2.76 Radius of G. –

It is denoted by r(G) and is defined by $r(G) = \min^m \{e(v): v \in V\}$.

Def. 2.77 Central point –

A point v is called central point if $r(G) = e(v)$ and the set of all central points is called centre of G.

2.71 Properties of trees.

We will try to establish relations between the number of edges and the vertices of various types in trees.

Property I 2.72 A tree with n-vertices has (n – 1) edges.

Proof –

We have applied induction method to prove it . It will be noted that for all the trees we will choose a root and consider the tree rooted.

Step I. When n = 1. In this case vertex has no edges and hence for n = 1 the statement is holds.

Step II. Assume that a tree with K vertices has (K – 1) edges ($K \in N$).

So, let a tree T has (k + 1) vertices and V is a leaf of T- (since tree is finite) and W – a parent of V. Removing from T the vertex V and the edge connecting w to v produces a tree T' with K-vertices

(Since the resulting graph is still connected and has no simple circuits). Hence using induction assumption T' has (K – 1) edges. This means that T' has K-edges as it has one more edge than T' (the edge connecting V and w). Hence the statement.

Property II 2.73 – A full m-ary tree with i (inter vertices) contains $n = m\,i + 1$ vertices.

Proof.

We know that every vertex with except the root, is the child of an internal vertex. Also, since each to the C (internal vertices) has m-children there are mi-vertex in the tree except the root. Hence the tree contains $n = mi + 1$ vertices.

2.74 Property-III A non-trival tree contains at least 2 vertices of degree1.

Proof.

A tree is called trivial if it has only one vertex.

Also a tree with n-vertices contains (n-1) edges . Since tree is a connected graph so no vertex has degree 0. Also, sum of degree of all vertices (S) = 2 (n-1).

Now if each of the n-vertices has degree equal to or greater than 2, then sum of degree of all vertices $s > 2n$ i.e. $2(n-1) > 2n$ (impossible). Hence at least one vertex has degree equal to one.

Property (iv) Prove that every vertex in a directed tree different from the root has a unique parent

Proof.

Take a vertex V in the directed free. Since V is not the root so there exists a a vertex V' having direct edge with V uniqueness of V' then, let V^{11} be the, another vertex such that V''V' is another edge. Since in a tree there can not exists another vertex V'' such that V'' V' is directed edge so even on the removal of V'' V' the graph will remain connected this means V' is unique parent of V.

2.73 Example :

If a tree has n- vertices of degree 1, 2- vertices of degree 2, 4-vetices of degree 3 and 3-vertices of degree 4. Then find n.

Solution —

If n_1 be the number of vertices and e be the number of edges in the tree. Then, S = Sum of degree of all vertices S = 2e and number of edges $e = (n_1 - 1)$

So, $S = 2(n_1 - 1) = n \times 1 + 2 \times 2 + 4 \times 3 + 3 \times 4 = n + 28$.

Also, total vertices $n_1 = n + 9$. But $S = 2(n_1 - 1) = 2(n + 9 - 1)$

Hence, $n + 28 = 2(n + 8) \Rightarrow n = 12$

So the tree has 12- vertices of degree one.

3.76 Property III A full-m-any tree with

(i) n vertices has $i = \dfrac{n-1}{m}$ internal vertices and

$$l = \frac{[(m-1)n+1]}{m} \text{ leaves.}$$

(ii) i - internal vertices has n = (mi + 1) vertices and

$l = (m - 1)\, i + 1$ leaves

(iii) l-leaves has $n = \dfrac{(ml-1)}{m-1}$ vertices and

$$i = \frac{(l-1)}{m-1} \text{ internal vertices.}$$

Proof –

As known, n = (mi + 1). Also, n = l + i,

because each vertex is either a leaf of an internal vertex.

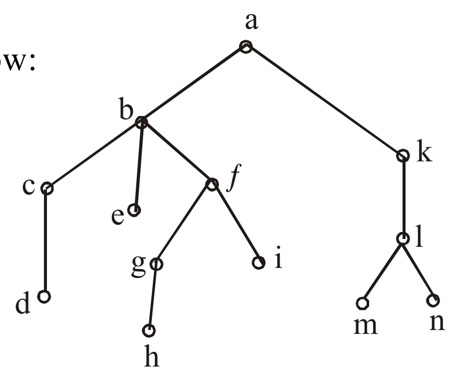
Now, $\dfrac{n-1}{m} = i$ and $l = n - i = n - \dfrac{n-1}{m}$

or, $l = \dfrac{mn-n+1}{m} = \dfrac{n(m-1)+1}{m}$ which proves (i) similarly, (ii) and (iii) can be proved.

At times we need to use rooted trees that are balanced so that the subtrees at each vertex contain path's of approximately the same length. The level of a vertex v in a rooted tree is the length of the unique path from the root to this vertex. The level of the root is defined to be zero. The height of a rooted tree is the maximum of the levels of the vertex or the height of a rooted tree is the length of the longest path from the root any vertex.

2.74 Example.

Find the height of the tree shown below:



Rooted Tree

Solution–

Hence a (root) is at level o vertices b, j, k are at level 1. vertices c, e, f, and l are at level 2. vertices d, g, i, m and n are at level 3. Lastly vertex h is at level 4. since the largest level of any vertex is 4 so height g there tree is 4.

A rooted m- ary tree of height h is balanced if all leaves are at levels h or (h – 1)

2.75 Example :

Which of the following trees are balanced ?

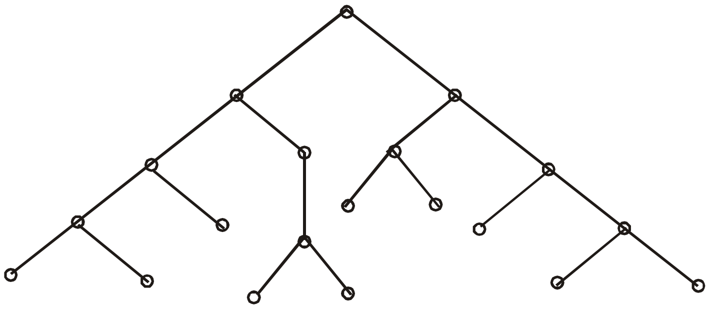


Fig. (1)

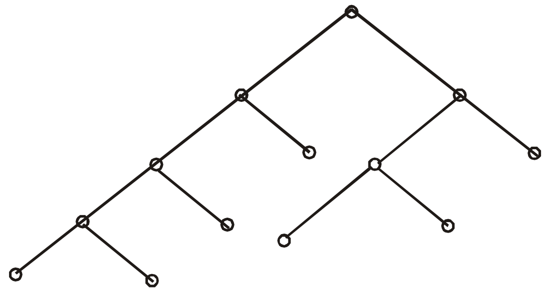


Fig. (2)

Solution –

Fig (i) tree is balanced. Because all its leaves are at levels at 3 and 4.

Fig (ii) tree is not balanced because it has leaves at levels 2, 3 and 4.

3.71 Proposition –

Let G be a graph then the following statements are equivalent :

(i) G is a tree.

(ii) G is connected and a cyclic (without cycle)

(iii) Between any two vertices of G there is precisely are path.

Proof.

(i) $\Rightarrow$(ii) This quite obvious because a cycle is a circuit.

(ii) $\Rightarrow$(iii) Let P_1 and P_2 be two different paths from vertex V to another vertex w. Then the closed walk from v to V obtained by following the vertices of P_1 and those of P_2 in reverse order would contain a cycle. (A construction of (ii)). Hence (ii) $\Rightarrow$ (iii)

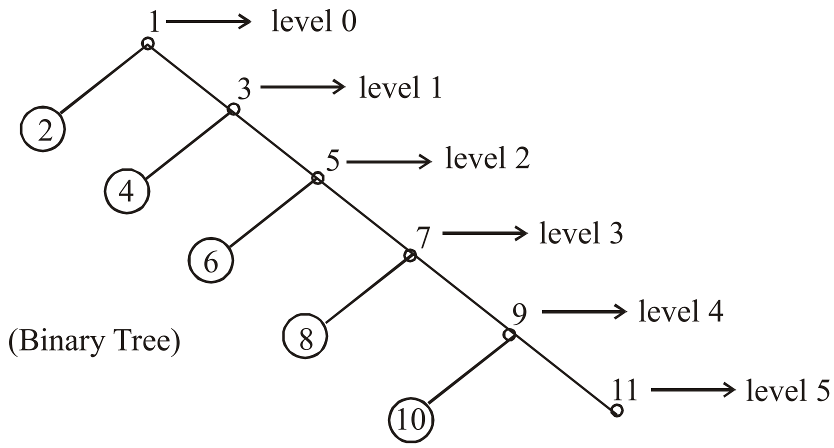
(iii) $\Rightarrow$ (i). Assume (iii) ie there is a path between any two vertices of G so G is connected Also, it cannot contain a circuit otherise it would contain a cycle and cycle determine two paths between any two vertices of it contradicting (iii).

Then (iii) $\Rightarrow$ (i) This proves the theorems.

2.78 Def. Height of a tree –

The maximum level of any vertex in a binary tree is called the height of the tree.

The number of vertices n is a binary tree is always odd. A non-perdant vertex in a tree is called an internal vertex in a tree is called an internal vertex.



$$\max l_{max} = \frac{11 - 1}{2} = 5$$

Theorem 2.72

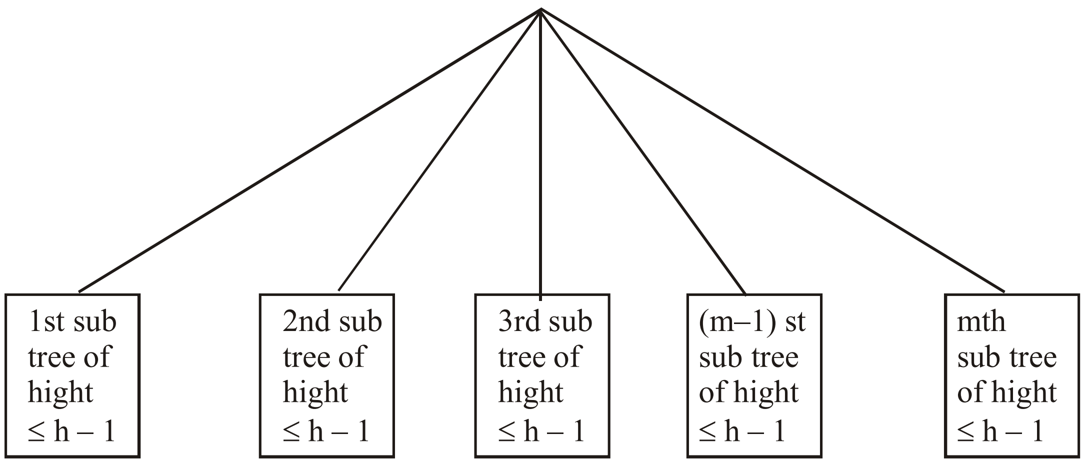
There are at most m^n-leaves in an m-ary tree of height h.

Proof.

The proof is based on induction method on height. Let consider m-ary tree of height one. These trees consist of a root with no more than m- childeren each of which is a leaf. Hence there are no more than $m^1 = m$ leaves in an m-ary tree of height 1. (True for h = 1).

Now assume that the result is true for all m-ary trees of height less than h.

Let T be an m-ary tree of height h. The leaves of T are the leaves of the subtrees of T obtained by deleting the edges from the root to each of the vertices at level 1.



Each of there subtrees has height less than or equal to (h - 1) So, using induction assumption, each of these rooted trees has at most m^{h-1} leaves. Since there are at most m such subtrees, each with a maximum of m^{h-1} leaves there are at most m. $m^{h-1} = m^h$ leaves in the rooted tree.

This proves the theorem.

2.73 Def.

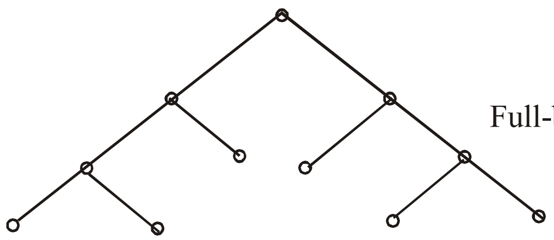
A tree is called full m-ary if every internal vertex has exactly m children. A binary tree is a m-ary tree with m-2.

Def. 2.74 Binary tree :

It is a noted, levelled tree in which any vertex has atmost two children.

Full binary tree is a tree in which every vertex has either two children or no children.

Thus a binary tree is a tree having exactly one vertex of degree two and the remaining vertices of degree one or three.

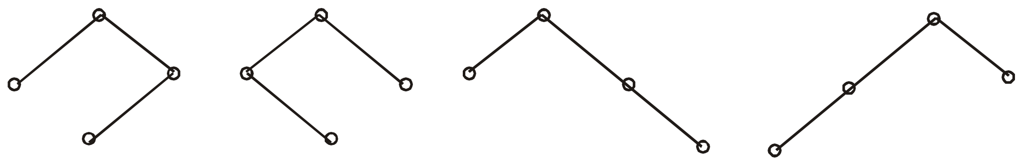


Full-binary tree

2.76 Example.

Draw all the distinct binary trees of 4-vertices.

Ans.



2.70 Trees as Models

Trees are used as models in diverse areas as computer science, botany, geology, chemistry, phychology etc.

2.71 (i) Saturated Hydrocarbons and Trees.

In course of enumerating the isomers of compounds of the from $C_n H_{2n+2}$ (Called saturated hydrocarbons), graphs were used to represent molecules. Here atoms were shown by vertices and bounds between then by edges.

Each carbon atom is represented by a vertex of degree 4 and each hydrogen atooms were represented by a vertex of degree one. There are (3n+2) vertices in a graph representing a compound of the form $C_n H_{2n+2}$. The number of edges in such a gaph is half the sum of the degrees of the vertices. Hence there are $\frac{(4n+2n+2)}{2}$ = (3n +1) edges in this graph. As the graph is connected and the number of edges is one less than the number of vertices, it must be a tree.

The tree for Butane is as follows :

```
      H
      |
  H – C – H
      |
  H – C – H
      |
  H – C – H
      |
  H – C – H
      |
      H
```

Theorem : 2.73

Prove that if G is a graph each of whose vertex has degree at least 2. then G contains a circuit.

Proof –

Let G be a graph and V be a vertex of G. Since deg $V_1 \neq 0$ So proceed to an adjacent vertex V_2. Since deg $V_2 \geq 2$, we may proceed to an adjacent vertex V_3 along a new edge. Clearly, $V_1V_2V_3$ is a path. In general, let $k \geq 3$ and that we have found a path through distinct vertices V_1, V_2, ... V_k. Since deg $V_k \geq 2$, V_k is adjacent to a vertex $V_{k-1} \neq V_{k+1}$. if $V_{k+1} \in \{V_1, V_2, ..., V_{k-2}\}$, We have a circuit : otherwise V_1V_2........ V_{k+1} is a path V_{k+1} Since there are only finitely many vertices in G, we cannot continue to find paths of increasing length. So evidently. we find a circuit.

In a trivial way a graph with a single vertex is a tree. A tree with more than one vertex can have no vertices of zero o. So all its vertices have degree at least 1. If they all had degree at least 2. there would be a circuit, thus we obtain a basic property of trees.

2.74 Theorem

An edge added to a tree must produce a circuit.

Proof :–

When an edge is added to a tree, the new graph remains connected and it is not a tree because it has an equal number of edges and vertices. So it must contain a circuit.

Definition 2.91 (Isomorphic) Two trees T_1 and T_2 having labeled vertices with the same set of labels are called isomorphic if and only if, for each pair of labels V and w, vertices V and w are adjacent in T_1 if and only if they are adjacent in T_2.

2.75 Theorem–

Prove that every vertex in a directed tree different from the root has a unique parent.

Proof.

Let V be a vertex in a directed tree. Let V is different from root. Then there exists a vertex V_1 form which there is a directed edge to V. This is unique V_1 is not unique. Then take another vertex V_2 such that V_2 V is another edge. obviously, the removal of V_2v will not disconnect the graph because it will be against the definition of the tree . Hence V_1 unique parent of V.

2.792 Spanning tree

A spanning tree of a connected graph G is a subgraph which is a tree and which includes every vertex of G.

If a graph has n-vertices, it can have n^{n-2} spanning tree.

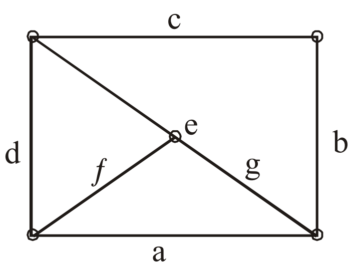
2.793 Def.

Minimum spanning tree – A minimum spanning tree of a weight graph is a spanning tree of least weight, that is a spanning for which the sum of the weights of all its edges is least among all spanning trees.

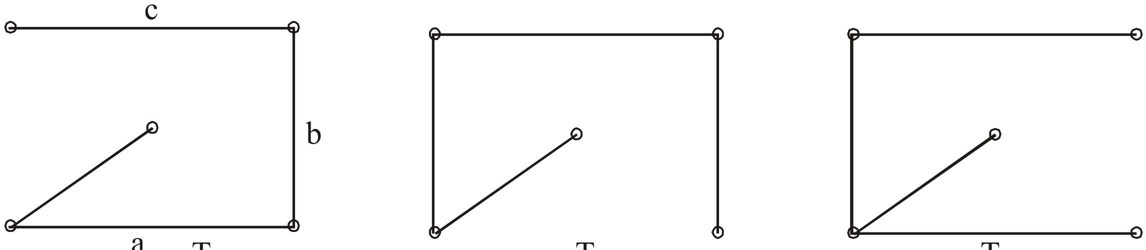
The concept of spanning tree exist only a connected graph G and if G has n-vertices, any spanning tree must necessarily contain (n – 1) edges.

It is not difficult to find a spanning tree in a connected graph G. If G has no circuits, then it is already a trees so G is a spanning tree for G.

If G contains a ciruit then we delete an edge so as to remove the circuit but leave the graph connected. By repeating this procedure, we eventually find a connected subgraph without circuits containing all the vertices of G i.e. a spanning tree.

Here is a graph G = 

and its three spanning trees



T_1 T_2 T_3

2.70 N.B. Every connected simple graph has a spanning tree.

2.794 Def.

Rooted spanning Tree. – It is a rooted tree containing edges of the gaph such that every vertex of the graph is an end point of one of the edges in the tree.

2.795 Def.

Spanning subgraph – A subgraph G which contains all the vertices of the original graph G is called spanning subgraph.

2.76 Theorem

A simple graph is connected if and only if it has a spanning tree.

Proof –

If part – Let a simple graph G has a spanning tree T. then T contains every vertex of G. Also the is a path in T between any two of its vertices. Since T is subgraphs of G, There is a path in G between any two of its vertices. Hence, G is connected.

Only if part. Assume that G is connected. Let G is not a tree then it must contain a simple circuit. we remove an edge from one of these simple circuits . Then the resulting graph will have one fewer edge but still contains all the vertices of G and is connected. This subgraph is still connected because when two vertices are connected by a path not containing this edge. we can construct such a path by interesting into the original path, at the point where the removes edge once was the simple circuit with this edge removed Now if this subgraph is not a tree, it has a simple circuit. So as before remove an edge that is in a simple circuit. Repeat this process untill no simple circuit remain. This is possible because there are only a finite number of edges in the graph. We will be see that the process will at one stage, terminate where no simple circuit remain. A tree is produced since the graph stays connected as edges are removed. obviously, this tree is a spanning tree since it contains every vertex of G.

N.B. 2.71 Spanning level is useful in networking.

2.77 Example.

There is a scheme to start chain letter. Each person who receives the letter is asked to send it on to four other people. Some people do not do this but others do not send any letter. How many people have seen the letter, including the first person, if no one receive more than one letter and if the chain letter ends alter there have been 100 people who need it but did not send it out ? How many people send out the letter.

Solution

The given chain problem can be reprosented using a 4-ary tree. The internal vertices correspond to people who send out the letter, and the leaves correspond to people who did not send it out. Since 100 people did not send out the letter, the number of leaves in this rooted tree is l = 100. Then using formula, n = number of people who have sent the letter =

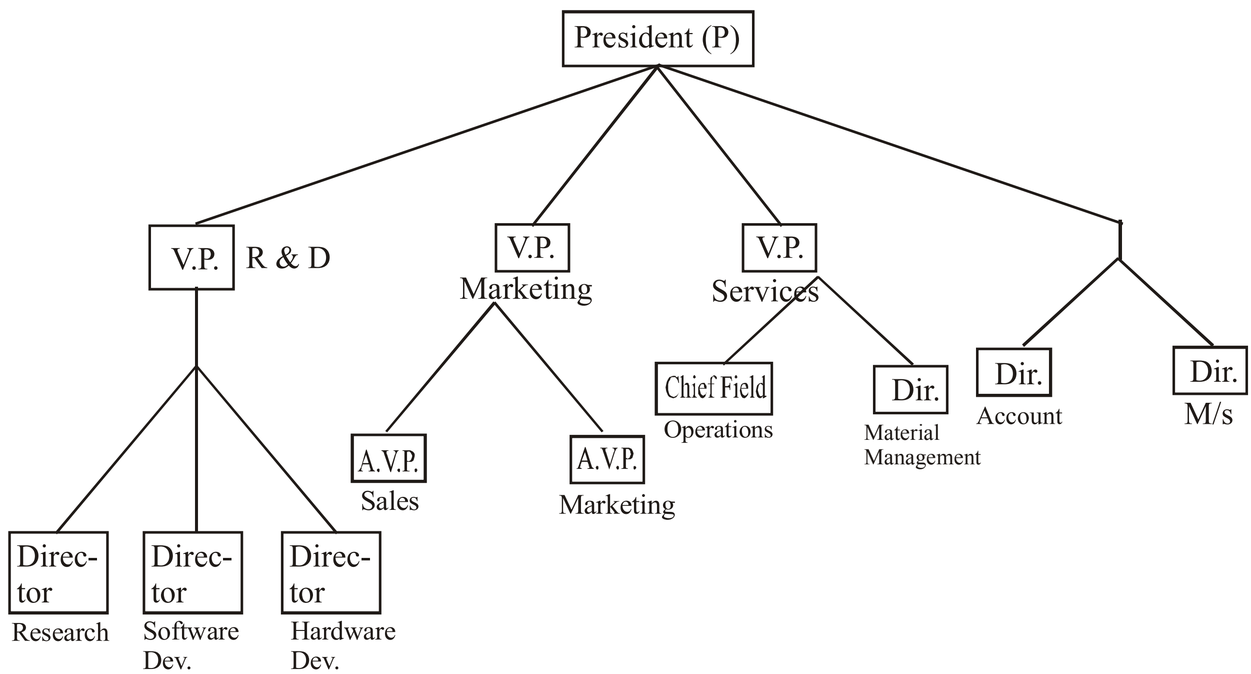
$\frac{(4.100-1)}{(4-1)}$ using formula $n = \frac{ml-1}{m-1}$ = 133.

The number of internal vertices is 133 – 100 = 33.

Hence 33 people sent out the letter.

2.78 Example Organisational representation

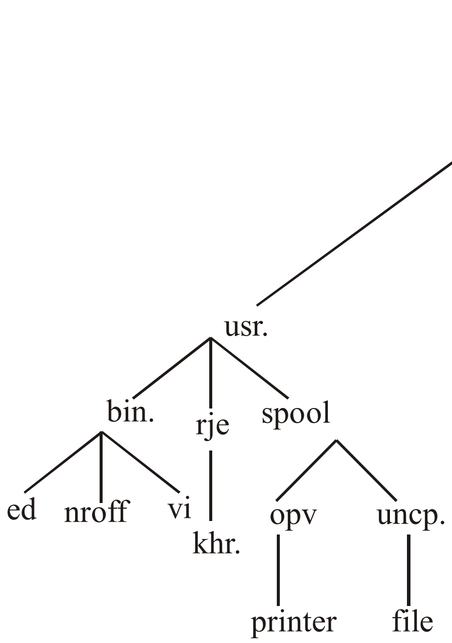
The structure of a big organization can be modeled using a rooted tree. In this tree the each vertex represents a position in the organization. An edge between two vertices indicates that the person represented by the initial vertex is the boss (direct) of the person repesented by the terminal vertex.



2.79 Example Computer file

A file system can be represented by a rooted tree. Files in a computer memory can be organized into directions. A directory can contain both files and subdirectaries. The file system is contained in root directory.

In this rooted tree root represents the root directory internal vertices represent sub directories and leaves represent ordinary files or empty directories.



2.8 Summary

This book is written to make some important mathematical ideas interesting and understable to the students of elementary level. The material in this text have included elementary terminologies and some important and useful concepts.

The first section deals with important definitions and properties.

In the second section we have discussed about subgraphs and induced graph. Sometimes, we need actually a part of a graph to solve the problem. when edges and vertices are removed from a graph, without removing end points of any remaining edges, a smaller graph is obtained. Such a graph is called a subgraph of the original graph. We have discussed paths, cycles and connectivity. Many problems can be modeled with paths formed by travelling along the edges of graphs. There are two notions of connectedness in directed graphs, depending on whether the directions of the edges are considered.

In the third section planar graphs. It gives a solution to the problem as to "Is it possible to draw $K_{3,3}$ in a plane. So that no two of its edges cross ? So we will discuss here whether a graph can be drawn in the plane without edges crossing. The house and utilition problem can give answer to it. There are many ways to represent a graph. When it is possible to find at least one way to represent this graph in a plane without any edge crossing ? A graph may be planar even if it is usually drawn with crossings, since it may be possible to draw it in a different way without crossing

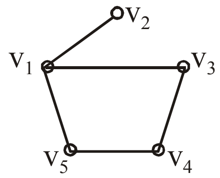
The fourth and the last section deals with the most commonly occuring kind of graph called a tree, perhaps because it can be drawn so that it looks a bit like an ordinary tree. Family tree are graphs that represent genealogical charts. Vertices represent the members of a family and edges represent parent child relationship. A tree cannot have a simple circuit, a tree cannot contain multiple edge or loops. Therefore, any tree must be a simple graph.

2.9 Model Questions

(1) What is the largest number of vertices in a graph with 35 edges if all vertices are of degree at least three ?

(2) Draw a graph with five vertices V_1, V_2, V_3, V_4, V_5 such that degree $V_1 = 3$, V_2 is an odd vertex deg $V_3 = 2$ and V_4 and V_5 are adjacent.

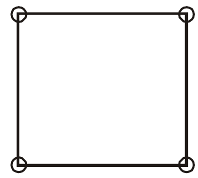
Ans.



(3) Suppose all vertices in a graph have odd degree K. Show that the total number of edges in a G is a multiple of K.

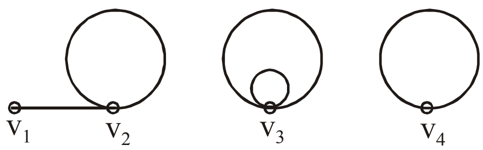
(4) Give example of a graph such that every vertex is adjacent to two vertices and any edge is adjacent to two edges.

Ans.



(5) Can there exist a graph with four vertices of degree 1, 2, 3, and 4, ?

(6) What are the degree of the vertices in the pseudograph ?



Ans. def.$(v_1) = 1$, deg$(v_2) = 3$

def$(v_3) = 4$, deg$(v_4) = 2$

(7) Prove that the graph which contains a triangle cannot be bipartite.

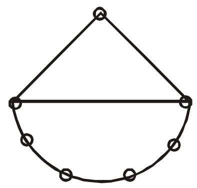
(8) Prove that the number of edges in a bipartite graph with n-vertices is at most $\frac{n^2}{4}$.

(9) Determine if there exists a graph whose degree sequence is the one specified below. In each case draw a graph if possible.

(i) 4, 4, 4, 3, 2

(ii) 5, 5, 4, 3, 2, 1

(iii) 1, 1, 1, 1, 1, 1.

(10) Is this graph  isomorphic to $K_{3,4}$?

(11) Explain why any graph is isomorphic to a subgraph of some complete graph.

(12) Prove that two graphs which are isomorphic must contain the same number of triangles.

(13) Find a connected graph with as few vertices as possible which has precisely two vertices of odd degree. (Ans. n is odd)

(14) For which values of $n > 1$, if any does K_n Eulerian ?

(15) Any closed walk in a graph contains a cycle. True or false.

(Ans – False.)

(16) A graph has 20 vertices. Any two distinct vertices x and y have the property that deg x + deg y $\geq$ 19. Prove that G is connected.

(17) How many isomers does pentane (C_5H_{12}) have ? Why. (Ans : 3)

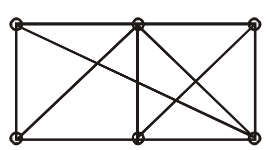
(18) Find a necessary and sufficient condition for a tree to be a complete bipartite graph.

(19) Prove that a tree a with two vertices of degree 3 must have at least four vertices of degree one.

(20) Draw all trees of n labeled vertices for n = 1, 2, 3, 4, 5

use formula :– The number of labeled trees with n vertices is n^{n-2}

(21) Show that the graph is planar by drawing an isomorophic plane graph with straight edges.



(22) If G is connected plane graph with $V \geq 3$ vertices and R regions, show that $R \leq 2V - 4$

(23) For which n, K_n is planar ? (Ans $n \leq 4$)

(24) Let G be a connected graph with V_1 vertices and E_1 edges and let H be a subgraph with V_2 vertices and E_2 edges. Show that $E_2 - V_2 \leq E_1 - V_1$

(25) Show that $K_{2,2}$ is homeomorphic to K_3.

2.91 References

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(4) Oystein ore – Graphs and Their uses.
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